EC9A0: Pre-sessional Advanced Mathematics Course Fixed Point Theorems

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University of Warwick, EC9A0: Pre-sessional Advanced Mathematics Course

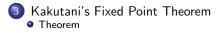
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Lecture Outline

Contraction Mapping Theorem

- Definition of Contraction
- BLACKWELL'S SUFFICIENT CONDITIONS
- Theorem
- Applications
- Brouwer's Fixed Point Theorem Definitions
 - Theorem



DEFINITION OF CONTRACTION

Definition

Let (X, d) be a metric space and $f : X \mapsto X$. f is a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$, $d(f(x), f(y)) \leq \beta d(x, y)$, $\forall x, y \in X$.

Example

Let $a, b \in \mathbb{R}$ with a < b, X = [a, b] and d(x, y) = |x - y|. Then f is a contraction if for some $\beta \in (0, 1)$,

$$rac{|f(x)-f(y)|}{|x-y|} \leq eta < 1$$
, for all $x,y \in X$ with $x
eq y$

That is, f is a contraction mapping if it is a function with slope uniformly less than one in absolute value.

BLACKWELL'S SUFFICIENT CONDITIONS

Theorem : Blackwell's sufficient conditions for a contraction

Let $X \subset \mathbb{R}^{K}$, and let B(X) be a space of bounded functions $f : X \mapsto \mathbb{R}$ with the sup norm. Let $T : B(X) \mapsto B(X)$ satisfy

• (monotonicity) $f, g \in B(X)$ and $f(x) \le g(x)$, for all $x \in X$, implies $(Tf)(x) \le (Tg)(x)$, for all $x \in X$;

(discounting) there exists some $\beta \in (0,1)$ such that $[T(f+a)](x) \le (Tf)(x) + \beta a$, for all $f \in B(X)$, $a \ge 0, x \in X$

where (f + a)(x) is the function defined by (f + a)(x) = f(x) + a.

Then T is a contraction with modulus β

THEOREM

CONTRACTION MAPPING THEOREM

Theorem

If (X, d) is a complete metric space and $T: X \mapsto X$ is a contraction mapping with modulus β , then

1 T has exactly one fixed point $x \in X$, and

2 for any $x_0 \in X$, $d(T^n x_0, x) \leq \beta^n d(x_0, x)$, n = 0, 1, 2, ...

Application I: Neoclassical Growth Model

Example

In the one sector optimal growth problem, an operator \mathcal{T} is defined by

$$(Tv)(x) = \max_{0 \le y \le f(x)} \{ U[f(x) - y] + \beta v(y) \}$$

• If $v(y) \le w(y)$ for all y, then $Tw \ge TV$ and so monotonicity holds.

• To show discounting note that:

$$T(v+a)(k) = \max_{\substack{0 \le y \le f(x) \\ 0 \le y \le f(x) }} \{ U[f(x) - y] + \beta[v(y) + a] \}$$

=
$$\max_{\substack{0 \le y \le f(x) \\ 0 \le y \le f(x) }} \{ U[f(x) - y] + \beta v(y) \} + \beta a$$

=
$$(Tv)(x) + \beta a$$

Application II: Differential Equations

Example

Consider the differential equation and boundary condition $\frac{dx(s)}{ds} = f[x(s)]$, all $s \ge 0$, with $x(0) = c \in \mathbb{R}$. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous, and for some B > 0 satisfies the Lipschitz condition $|f(a) - f(b)| \le B|a - b|$, all $a, b \in \mathbb{R}$. For any t > 0, consider C[0, t], the space of bounded continuous functions on [0, t], with the sup norm.

$$(Tv)(s) = c + \int_0^s f[v(s)]dz, 0 \le s \le t$$

maps C[0, t] into itself.

- **2** Show that for some $\tau > 0$, T is a contraction on $C[0, \tau]$.
- Show that the unique fixed point of T on $C[0, \tau]$ is a differentiable function, and hence that it is the unique solution on $[0, \tau]$ to the given differential equation.

DEFINITIONS

- f maps the set $X \subset \mathbb{R}^K$ into itself if $f(x) \in X$ for all $x \in X$.
- We would like to find conditions ensuring that any continuous function mapping X into itself has a fixed point.
- The following example shows that some restrictions must be placed on X:
 - f(x) = x + 1 maps $\mathbb R$ into itself.
 - f(x) has no fixed point since f(x) = x implies 1 + x = x, an absurd.

BROUWER'S FIXED POINT THEOREM

Theorem L.E.J. Brouwer's fixed point theorem

Let X be a nonempty compact convex set in \mathbb{R}^{K} , and f be a continuous function mapping X into itself. Then f has a fixed point x^* .

- For $X = \mathbb{R}$, a nonempty compact convex set is a closed interval [a, b].
- A continuous function $f : [a, b] \mapsto [a, b]$ must have a fixed point.
- This follows from the IVT:
 - Define g(x) = f(x) x.
 - x is a fixed point of f if and only if g(x) = 0.
 - Since $g(a) \ge 0$ and $g(b) \le 0$, there is $x^* \in [a, b]$ such that $g(x^*) = 0$.
- We use Brouwer's fixed point Theorem to prove existence of equilibrium in a pure exchange economy.

Kakutani's Fixed Point Theorem

- Brouwer's Theorem deals with fixed points of continuous functions.
- Kakutani's theorem generalises the theorem to correspondences.

Definition

An element $x \in X$ is a fixed point of a correspondence $F : X \mapsto X$ if $x \in F(x)$.

Theorem Kakutani's Fixed Point Theorem

Let X be a nonempty compact convex set in \mathbb{R}^{K} and $F : X \mapsto X$ be a correspondence. Suppose that:

- **(**) F(x) is a nonempty convex set in X for each $x \in X$
- **2** *F* is upper hemicontinuous.

Then F has a fixed point x^* in X

• We use Kakutani's Fixed Point Theorem to prove existence of a Mixed Strategy Nash Equilibrium in an N-player game with finite (pure) strategy sets.

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