Abstract

Tarski's theorem advises us that no completely satisfactory definition of the term *true sentence* can be expected. It is here shown that, nevertheless, it is possible to formulate within a fragment $\mathbb{T}$ of Zermelo set theory $\mathbb{Z}$ a definition of the truth of sentences that is *materially adequate*, *formally correct*, *explicit*, *universal*, *versatile*, and *modestly paraconsistent*. The definition is extended from sentences to deductive theories, and an investigation is begun into why the antinomy of the liar fails to arise.

Truth Defined

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1 Introduction

It is the purpose of this paper to offer an explicit and materially adequate definition of truth for any denumerable language that contains, in addition to the standard elementary theory of identity, enough of its own syntax to name all its expressions and to distinguish among them effectively. The languages for which the definition is appropriate include the language of Zermelo–Fraenkel set theory $\mathbb{ZF}$, and any applied language $\mathcal{L}^+$ constructed from it by adding items of extralogical vocabulary. The *background theory* $\mathbb{T}$ of the investigation, which is fixed, is a fragment of Zermelo set theory $\mathbb{Z}$. A detailed discussion and defence of the definition are offered in Miller (2009).

A *structural-descriptive name* $\ulcorner y \urcorner$ of the formula y is a name, such as a quotation name or a numeral for the Gödel number of y, from which the syntactic structure of y may be deduced. The most important demand imposed on any structural-descriptive names $\ulcorner x \urcorner$ and $\ulcorner z \urcorner$ given to the formulas x, z, is that the sentence $\ulcorner x \urcorner = \ulcorner z \urcorner$ (that is, the sentence consisting of $\ulcorner x \urcorner$, the identity sign, and

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The two papers are dedicated to the memory of Karl Popper, whose devotion to truth never wavered.

$\ulcorner z \urcorner$, in that order) be demonstrable or refutable in the background theory $\mathbb{T}$.

Tarski called a definition of the term 'true sentence' *materially adequate* for the language $\mathcal{L}$ if within the background theory $\mathbb{T}$ it logically implies every instance of the **T**-scheme

$$P \text{ is a true sentence if \& only if } p, \qquad (\mathsf{T})$$

where 'p' is replaced by a sentence y of $\mathcal{L}$ and 'P' is replaced by a structural-descriptive name $\ulcorner y \urcorner$ of y. Tarski observed that for a language $\mathcal{L}$ with only finitely many distinct sentences $x_0, \ldots, x_{j-1}$, the problem of the definition of truth, as he conceived it, that is, the problem of providing a materially adequate definition of the term 'true sentence', can be fully solved, provided that each sentence x_i of $\mathcal{L}$ is furnished with a structural-descriptive name $\ulcorner x_i \urcorner$ (Tarski 1933a, p. 188). We may define 'y is a true sentence', or $\mathsf{Tr}(y)$, by either of the equivalences

$$\mathsf{Tr}(y) \quad \Leftrightarrow_{\text{Df}} \quad (y = P_0 \ \wedge \ p_0) \vee \ \ldots \vee \ (y = P_{j-1} \ \wedge \ p_{j-1}), \qquad (1.0)$$

$$\mathsf{Tr}(y) \quad \Leftrightarrow_{\text{Df}} \quad (y = P_0 \ \rightarrow p_0) \ \wedge \ \ldots \ \wedge \ (y = P_{j-1} \ \rightarrow p_{j-1}), \qquad (1.1)$$

where, for each $i < j$, the sentence x_i replaces 'p_i', and the name $\ulcorner x_i \urcorner$ replaces 'P_i'. It is evident that, provided that the connectives $\vee$, $\wedge$, and $\rightarrow$ are governed by the usual (classical) rules, both (1.0) and (1.1) are materially adequate definitions of truth. It is evident too that each of these definitions is longer than any sentence of the language $\mathcal{L}$, and has to be formulated outside $\mathcal{L}$.

It is the burden of *Tarski's theorem* that no language $\mathcal{L}$ whose deductive structure is of a complexity that allows forms of self-ascription can consistently include within itself a definition of the term 'true sentence' that is materially adequate for $\mathcal{L}$. Languages to which Tarski's theorem applies include in particular the languages of elementary arithmetic and set theory.

Tarski's principal achievement was to show how, for each elementary language $\mathcal{L}_0$, it is possible to prepare, within a richer *metalanguage* $\mathcal{L}_1$, an explicit definition, materially adequate for $\mathcal{L}_0$, of the term 'true sentence'. The metalanguage $\mathcal{L}_1$ must be richer than $\mathcal{L}_0$ in the sense that it must incorporate stronger set-theoretical postulates than $\mathcal{L}_0$ does, not in the sense that it must boast a wider vocabulary. To generate a definition of the term 'true sentence' that is materially adequate for $\mathcal{L}_1$, a yet richer metametalanguage $\mathcal{L}_2$ must be presumed on. An unending hierarchy of languages is in this way initiated, and the concept of 'true sentence' is never completely defined.

Tarski's theorem will not be contested here. But the conclusion customarily drawn from it, that there can be no fully satisfactory definition of the concept of truth, for arithmetic or for set theory, will be constructively contested (if not refuted). Truth, as it is to be defined in this paper, is not a property Tr of sentences but a function $\boldsymbol{\theta}$, from terms (names, descriptions, variables) that stand for sentences, to deductive theories. Described informally, $\boldsymbol{\theta}(\mathfrak{y})$, *the truth of* $\mathfrak{y}$, is the theory that states the conditions under which the sentence named by the term $\mathfrak{y}$ is true. In particular, if x is any sentence of the extended language $\mathcal{L}^+$, and $\ulcorner x \urcorner$ is a structural-descriptive name of x, then $\boldsymbol{\theta}(\ulcorner x \urcorner)$ turns out to be identical with the theory $\mathsf{Cn}(\{x\})$ that consists of all the logical consequences of x; that is to say, $\boldsymbol{\theta}(\ulcorner x \urcorner)$ and x are logically interderivable. The definition of truth to be presented is materially adequate for the entire language $\mathcal{L}^+$ of applied set theory.

For most of the paper, a somewhat different function, $\boldsymbol{\vartheta}$, will be investigated alongside the function $\boldsymbol{\theta}$. Only in § 7 will a genuine preference be expressed between them as definitions of truth. Everything said in the previous paragraph concerning $\boldsymbol{\theta}$ applies equally to the alternative definiens for truth, $\boldsymbol{\vartheta}$.

2 Deductive Theories

This section summarizes the principal results of Tarski's *general metamathematics* (or *calculus of deductive systems*) that are initially required for an understanding of the definition of truth to be presented in § 3 below. These results, together with the less ordinary results presented in § 4 below, can be found, in most cases without proofs, in Tarski (1935–1936). In addition to the transposition (now common) of Tarski's original connotations of the terms 'deductive theory' and 'deductive system', two variations deserve remark. The first is that we shall allow sets of well-formed formulas, and not only sets of sentences (closed formulas), to count as deductive theories. This change makes little difference until we come in § 7 to define the truth of theories themselves. The second variation is that we shall invert the ordering on theories (by set-theoretical inclusion) that Tarski uses throughout his works, and use instead the ordering by logical derivability. Where Tarski writes $\mathsf{X} \subseteq \mathsf{Z}$, we shall often write $\mathsf{Z} \vdash \mathsf{X}$; the operations of product (intersection) and sum in Tarski's treatment, both finite and infinite, become disjunction and conjunction in the treatment here, with the awkward repercussion that a set-theoretically expanding sequence of theories will here be called *decreasing* (see Theorem 4 below). A more significant discrepancy is that whereas the complement $\bar{\mathsf{Y}}$ that Tarski defines for the theory Y is a pseudocomplement, which obeys the law of non-contradiction but not the law of excluded middle, the complement Y' that we shall define (in 4.3) is an authocomplement (that is, a dual pseudocomplement), which obeys the law of excluded middle but not the law of non-contradiction. This all needs to be said explicitly lest anyone should compare the present report with Tarski (1935–1936) and conclude with a sigh that everything here is upside down. It is intended that everything be upside down.

Let S be the set of formulas of a denumerable language $\mathcal{L} \subseteq \mathcal{L}^+$. An operation $\mathsf{Cn} : \wp(S) \mapsto \wp(S)$ is a *consequence operation* if it fulfils the conditions of *idempotence* (2.0), and of *finitariness* (2.1):

$$X \subseteq S \implies X \subseteq \mathsf{Cn}(X) = \mathsf{Cn}(\mathsf{Cn}(X)) \subseteq S \tag{2.0}$$

$$X \subseteq S \implies \mathsf{Cn}(X) = \bigcup\{\mathsf{Cn}(Y) \mid Y \subseteq X \text{ and } |Y| < \aleph_0\}. \tag{2.1}$$

By (2.1), the operation Cn must also be monotone:

$$X \subseteq Z \subseteq S \implies \mathsf{Cn}(X) \subseteq \mathsf{Cn}(Z). \tag{2.2}$$

The pair $\langle S, \mathsf{Cn}\rangle$ may sometimes be called a *deductive system* or (to use an expression of Tarski 1930b, Introduction) a *deductive discipline*, or simply a *logic*. Reference to the set S will usually be omitted. With each consequence operation Cn is associated a *derivability* relation $\vdash$ such that $y \in \mathsf{Cn}(Y)$ if & only if $Y \vdash y$. The derivability notation will often, for the sake only of flexibility, be read 'Y

implies y'. Note that the logic $\mathbf{Cn}$ need not be identical with the logic that regulates the background theory $\mathbb{T}$. It may be stronger, it may be weaker. Explicit assumptions about $\mathbf{Cn}$ that will made in the course of the paper include: (i) the conjunction $x \wedge z$ of two formulas is always defined; (ii) their disjunction $x \vee z$ is also defined; (iii) both are defined, and the distributive law (2.7) and its dual hold for all formulas; (iv) the conditional $x \to z$ is defined for some or all pairs of formulas (if it is defined for all pairs, then the distributive law (2.7) holds); (v) $\mathbf{Cn}$ includes at least the whole of classical sentential logic. When we come, in § 3, to define truth, it will be mandatory that (vi) $\mathbf{Cn}$ includes, in addition to the standard logic of identity, structural-descriptive names for all formulas in S (and the capacity to prove $\ulcorner x \urcorner \neq \ulcorner z \urcorner$ whenever x and z are distinct formulas), not forgetting a rule of substitution for free variables, so that, for example, if y and w are free then $\mathbf{Cn}(y \neq w)$ contains each identity of the form $\ulcorner x \urcorner \neq \ulcorner z \urcorner$. In §§ 5f. we shall consider what happens when (vii) the logic $\mathbf{Cn}$ is, like elementary Peano arithmetic and $\mathbb{ZF}$, both ω-consistent and incompletable. It is easily proved in $\mathbb{T}$ that if $\mathbf{Cn}$ is a logic, and $X \subseteq S$, then $\mathbf{Cn}_X(Y) = \mathbf{Cn}(X \cup Y)$ is also a logic.

If $\mathbf{Y} = \mathbf{Cn}(\mathbf{Y})$, the set $\mathbf{Y} \subseteq S$ is a (deductive) *theory*. It is a (finitely) *axiomatizable* theory if $\mathbf{Y} = \mathbf{Cn}(Y)$ for some finite $Y \subseteq S$, in which case we may write $\mathbf{y}$ in place of $\mathbf{Y}$. In classical logic, where the operation $\wedge$ of conjunction exists and, moreover, $\mathbf{Cn}(\varnothing) = \mathbf{Cn}(\top)$ for any theorem $\top$, finite axiomatizability is the same as axiomatizability by a single formula: $\mathbf{Cn}(\mathbf{Y}) = \mathbf{Cn}(\{y\})$. It is customary (and does no harm) to identify formulas x, z for which $\mathbf{Cn}(\{x\}) = \mathbf{Cn}(\{z\})$, and to identify the formula y with the axiomatizable theory $\mathbf{Cn}(\{y\})$, which is more commonly written $\mathbf{Cn}(y)$. Such an identification is innocuous in regard to (finite) axiomatizability: $\mathbf{Cn}(\mathbf{Y}) = \mathbf{Cn}(Y)$ for a finite set Y of formulas if & only if $\mathbf{Cn}(\mathbf{Y}) = \mathbf{Cn}(\bigcup \mathcal{Y})$, where $\mathcal{Y}$ is a finite set, for example $\{\mathbf{Cn}(y) \mid y \in Y\}$, of finitely axiomatizable theories. Tarski's calculus of deductive theories is of independent interest only when the deductive discipline $\langle S, \mathbf{Cn} \rangle$ is *non-trivial* in the sense that, for each formula $x \in S$, there are infinitely many formulas z for which $\mathbf{Cn}(x) \neq \mathbf{Cn}(z)$.

As earlier noted, $\mathbf{Z} \vdash \mathbf{X}$ is defined to mean $\mathbf{X} \subseteq \mathbf{Z}$. It must be borne in mind that this extended derivability relation $\vdash$ is not finitary; for evidently, $\mathbf{X} \vdash \mathbf{X}$ does not imply that there is a finite subset $\mathbf{Y} \subseteq \mathbf{X}$ for which $\mathbf{Y} \vdash \mathbf{X}$. A theory $\mathbf{Y}$ is *consistent* provided that $\mathbf{Y} \neq S$, and is *maximal* (or *complete*) if it is consistent and has no consistent proper extension. Lindenbaum's theorem (Tarski 1935–1936, p. 366), whose proof requires of the logic $\mathbf{Cn}$ only that $\mathbf{Cn}(S)$ be axiomatizable, states that if $\mathbf{Y}$ is a consistent theory then it has at least one maximal extension $\boldsymbol{\Omega}$.

The *disjunction* $\mathbf{X} \vee \mathbf{Z}$ and *conjunction* $\mathbf{X} \wedge \mathbf{Z}$ of the theories $\mathbf{X}$ and $\mathbf{Z}$ are defined like this:

$$\mathbf{X} \vee \mathbf{Z} \quad = \quad \mathbf{X} \cap \mathbf{Z}, \tag{2.3}$$

$$\mathbf{X} \wedge \mathbf{Z} \quad = \quad \mathbf{Cn}(\mathbf{X} \cup \mathbf{Z}). \tag{2.4}$$

Proving that $\mathbf{X} \vee \mathbf{Z}$ is a theory requires little work; proving that $\mathbf{X} \wedge \mathbf{Z}$ is a theory requires none. The *join* $\bigvee \mathcal{K}$ and the *meet* $\bigwedge \mathcal{K}$ of a family $\mathcal{K}$ of theories are defined in a similar way:

$$\bigvee \mathcal{K} \quad = \quad \bigcap \mathcal{K}, \tag{2.5}$$

$$\bigwedge\mathcal{K} \quad = \quad \mathbf{Cn}(\bigcup\mathcal{K}). \tag{2.6}$$

If $\mathcal{T}$ is the class of all deductive theories in $\mathcal{L}$, then the meet $\bigwedge\mathcal{T}$ is identical with the set S of all formulas and is itself a theory. To emphasize this, we sometimes write $\mathbf{S}$ instead of S. Under the derivability relation $\vdash$ associated with $\mathbf{Cn}$, the *absurd* theory $\mathbf{S}$ is the logically strongest of all, while the weakest of all is (by (2.2)) the *trivial* theory $\mathbf{L}$, which is identical with $\bigvee\mathcal{T}$ or $\mathbf{Cn}(\varnothing)$. $\mathbf{L}$ is always axiomatizable. Note that the disjunction $\mathbf{X} \vee \mathbf{Z}$ and conjunction $\mathbf{X} \wedge \mathbf{Z}$ of two theories are well defined even if there exist no operations of disjunction $x \vee z$ and conjunction $x \wedge z$ on formulas, and the theories $\mathbf{S}$ and $\mathbf{L}$ are well defined even if the consequence operation $\mathbf{Cn}$ recognizes no minimal and maximal formulas $\bot$ and $\top$. Even when the consequence operation $\mathbf{Cn}$ does not support $\vee$ and $\wedge$, we shall freely mix in a single expression terms for formulas and terms for theories; $x \vee \mathbf{X}$, for example, is to be regarded as shorthand for $\mathbf{Cn}(x) \vee \mathbf{X}$.

It is a routine task to verify that if the logic $\mathbf{Cn}$ includes the standard elimination and introduction rules for disjunction $\vee$ and conjunction $\wedge$, and in addition the distributive law

$$y \vee (x \wedge z) \quad = \quad (y \vee x) \wedge (y \vee z) \tag{2.7}$$

(or its dual, which is equivalent) holds for all formulas x, y, z, then the class $\mathcal{T}$ of all theories forms a complete distributive lattice with join $\vee$ and meet $\wedge$ defined as in (2.3) and (2.4). We have in particular the absorption laws (2.8) and the distributive law (2.9) for theories:

$$\mathbf{X} \vee (\mathbf{X} \wedge \mathbf{Z}) \quad = \quad \mathbf{X} \quad = \quad (\mathbf{X} \vee \mathbf{Z}) \wedge \mathbf{X}, \tag{2.8}$$

$$\mathbf{Y} \vee (\mathbf{X} \wedge \mathbf{Z}) \quad = \quad (\mathbf{Y} \vee \mathbf{X}) \wedge (\mathbf{Y} \vee \mathbf{Z}). \tag{2.9}$$

Only a handful of the many possible distributive laws are valid for infinite joins and meets, even if $\mathbf{Cn}$ is a classical consequence operation. If the distributive law (2.7) holds for $\mathbf{Cn}$, then

$$\bigwedge\{\mathbf{X} \vee \mathbf{Z} \mid \mathbf{Z} \in \mathcal{K}\} \quad = \quad \mathbf{X} \vee \bigwedge\mathcal{K}, \tag{2.10}$$

$$\bigvee\{\mathbf{X} \wedge \mathbf{Z} \mid \mathbf{Z} \in \mathcal{K}\} \quad \vdash \quad \mathbf{X} \wedge \bigvee\mathcal{K}, \tag{2.11}$$

with equality in (2.11) if the family $\mathcal{K}$ is finite. If in addition, the consequence operation $\mathbf{Cn}$ admits a conditional $x \to z$ for each two formulas x, z (in which case (2.9) holds too) then:

$$\bigvee\{\mathbf{x} \wedge \mathbf{Z} \mid \mathbf{Z} \in \mathcal{K}\} \quad = \quad \mathbf{x} \wedge \bigvee\mathcal{K}. \tag{2.12}$$

It deserves to be noted that although the meet $\bigwedge\mathcal{K}$ behaves very much as we expect an infinite conjunction to behave, the join $\bigvee\mathcal{K}$ is somewhat anomalous in that it can hold ('be true' intuitively) even if no element of $\mathcal{K}$ holds. This oddity may be illustrated by the language of classical elementary logic with identity and no other predicates or individual names. In this language we may, for every $j > 0$, formulate a sentence ω_j that states that the universe contains exactly j elements. Each theory $\boldsymbol{\omega}_j = \mathbf{Cn}(\omega_j)$ is consistent and categorical, and hence maximal. The theory $\mathbf{\Omega} = \bigwedge\{\mathsf{Cn}(\neg\omega_j) \mid j > 0\}$, which states that the universe is infinite, is consistent (by (2.1)) and maximal (by Vaught's test). If $\boldsymbol{\omega}_j \vdash y$ for every $j > 0$, then $\neg y \vdash \neg\omega_j$ for every j, and so $\neg y \vdash \mathbf{\Omega}$. Since $\mathbf{\Omega}$ is maximal, and not axiomatizable (as shown in Corollary 6 below), $\neg y$ is inconsistent. In other words, y is a theorem, and the join $\bigvee\{\boldsymbol{\omega}_j \mid j > 0\}$ is the trivially true theory L, despite its not exhausting all the possibilities.

More advanced aspects of Tarski's calculus of deductive theories are expounded in § 4 below.

3 The Definition of Truth

We henceforth identify expressions of the language $\mathcal{L}^+$ with their Gödel numbers, or other set-theoretical codes, with the result that all expressions are sets, and all variables are variables for sets; that is, they may be replaced by names of sets. For ease of exposition we shall often use the locution 'name of' rather loosely in place of the perhaps more correct words 'expression, constant or variable, standing for', so that a variable (such as $\mathfrak{y}$ below) that may be replaced by a name (such as $\ulcorner 0 = 1 \urcorner$) of a formula may also be called a name of a formula. We may reasonably suppose that the set $\mathfrak{N}$ of all names of sets is definable in the background theory $\mathbb{T}$.

As announced earlier, it will be assumed that the logic embodied in the operation Cn includes the standard elementary logic of identity. We shall use $=$ and $\neq$ as names of the equality and the inequality signs (and later $\in$ to name the membership sign). Structural-descriptive names for all elements of S must also be available, and $\ulcorner x \urcorner \neq \ulcorner z \urcorner$ must be demonstrable (that is, $\ulcorner x \urcorner \neq \ulcorner z \urcorner \in \mathsf{L}$) whenever x and z are distinct formulas in S, and refutable whenever $x = z$. The extent to which the logic Cn needs to include also some sentential or quantificational logic is left open. The background theory $\mathbb{T}$, which is a fragment of the theory $\mathbb{Z}$, is assumed to contain as much elementary logic as it needs.

In the remainder of the paper we shall, for the sake of clarity, adopt '$\mathfrak{x}$', '$\mathfrak{y}$', '$\mathfrak{z}$', ... as variables for names of formulas, open to substitution, and revert to 'u', 'v', 'w', 'x', 'y', 'z', ... as variables for formulas. As is usual in formal work, we shall rarely, outside this paragraph, display any of the symbols of which we speak, but only their names. If α and γ are strings of symbols, then we shall use '$\alpha\gamma$' as a name for the concatenation $\alpha\frown\gamma$ (in that order) of α and γ.

We now proceed rigorously to generalize the right-hand sides of the two equivalent finite definitions (1.0) and (1.1), by replacing the finite disjunction in (1.0), for example, by a generalized join. We may define a function $\boldsymbol{\vartheta} : \mathfrak{N} \mapsto \mathcal{T}$ from expressions for formulas to deductive theories

$$\boldsymbol{\vartheta}(\mathfrak{y}) \quad =_{\mathrm{Df}} \quad \bigvee\{\mathsf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \ \wedge \ y) \mid y \in S\}. \tag{3.0}$$

In the same way we may generalize the alternative definition (1.1) of '$\mathsf{Tr}(y)$' in the finite case:

$$\boldsymbol{\theta}(\mathfrak{y}) \quad =_{\mathrm{Df}} \quad \bigwedge\{\mathsf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \rightarrow y) \mid y \in S\}. \tag{3.1}$$

It will be helpful later, especially in § 5, to have the definition of the function:

$$\boldsymbol{\delta}(\mathfrak{y}) \quad =_{\mathrm{Df}} \quad \bigvee\{\mathsf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\}. \tag{3.2}$$

The aim of this paper is to give a definition of truth that requires as little deviation from classical logic as possible. We have implicitly eschewed recourse to infinitary logic, and now we eschew recourse to intensional logic. It is evident that if the definiendum of the functions $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$ (more exactly, of the functors '$\boldsymbol{\vartheta}$' and '$\boldsymbol{\theta}$') are to be names of deductive systems, then the substituends for $\mathfrak{y}$ in (3.0) and (3.1) must be not formulas but expressions standing for formulas.

Provided that due caution is exercised, however, both '$\boldsymbol{\vartheta}(\mathfrak{y})$' and '$\boldsymbol{\theta}(\mathfrak{y})$' may be read intensionally as 'the *semantical value* of the formula (named by) $\mathfrak{y}$'; and when $\mathfrak{y}$ is the name of a sentence, both '$\boldsymbol{\vartheta}(\mathfrak{y})$' and '$\boldsymbol{\theta}(\mathfrak{y})$' may be read also as 'the truth of the sentence (named by) $\mathfrak{y}$'. In at least two places it will be rewarding to have the functions $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$ defined for all formulas: in § 7, where we shall discuss the truth of theories, and in Miller (2009), where the compositionality of the semantic values will be investigated. Speaking very roughly, we should like it to be the case that the semantic value of a quantified sentence is in some way related to the semantic value of the open formula within the scope of the quantifier. These matters will be made properly precise in the appropriate places.

Theorem 1 Both (3.0) and (3.1) are materially adequate definitions of truth; that is,

$$\boldsymbol{\vartheta}(\ulcorner x \urcorner) = \quad \mathbf{Cn}(x) \quad = \boldsymbol{\theta}(\ulcorner x \urcorner) \tag{3.3}$$

whenever $\ulcorner x \urcorner$ is the structural-descriptive name of a formula $x \in S$.

Proof If $\mathfrak{y}$ is replaced by a structural-descriptive name $\ulcorner x \urcorner$, then $\mathfrak{y} = \ulcorner y \urcorner$ is either demonstrable or refutable in the logic $\mathbf{Cn}$, and hence the term $\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \wedge y)$ is either $\mathbf{Cn}(x)$ or $\mathbf{S}$, and $\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to y)$ is either $\mathbf{Cn}(x)$ or $\mathbf{L}$. We conclude that $\boldsymbol{\vartheta}(\ulcorner x \urcorner) = \mathbf{Cn}(x) = \boldsymbol{\theta}(\ulcorner x \urcorner)$. ■

For a defence of the assumption here that material adequacy is rendered as well in terms of interderivability (or equivalence) as in terms of the usual biconditionals, see Miller (2009).

Theorem 2 If the formula named by $\mathfrak{x}$ is implied in $\mathbf{Cn}$ by the formulas with names $\{\mathfrak{z}_i \mid i < k\}$, then $\boldsymbol{\vartheta}(\mathfrak{x}) \subseteq \bigwedge\{\boldsymbol{\vartheta}(\mathfrak{z}_i) \mid i < k\}$.

Proof Immediate using (2.6). ■

This theorem may be taken to say (in a slightly odd way) that in a valid inference the truth of the premises is transferred to the truth of the conclusion. A similar result holds for $\boldsymbol{\theta}$.

We now show that $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$ may not be the same function, even if S is finite. This difference becomes evident if for $\mathfrak{y}$ is substituted a term $\mathfrak{v}$ that demonstrably names no formula $y \in S$, for then $\boldsymbol{\vartheta}(\mathfrak{v})$ is, by (3.0), the join of theories all identical with $\mathbf{S}$, while $\boldsymbol{\theta}(\mathfrak{v})$ is, by (3.1), the meet of theories all identical with $\mathbf{L}$. In short, $\boldsymbol{\vartheta}(\mathfrak{v}) = \mathbf{S} \neq \mathbf{L} = \boldsymbol{\theta}(\mathfrak{v})$. The theory $\boldsymbol{\delta}(\mathfrak{v})$, which expresses something like the idea that $\mathfrak{v}$ demonstrably names some formula of S, is also identical with $\mathbf{S}$. The divergence between $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$ is, however, sometimes more dramatic than this. We begin with a lemma that is a kind of generalization of the rules for the conditional.

Theorem 3 If the logic **Cn** constrains enough conditional sentences into obeying both the rule of modus ponens and the rule of conditional proof, then

$$\boldsymbol{\vartheta}(\mathfrak{y}) \quad = \quad \boldsymbol{\theta}(\mathfrak{y}) \wedge \boldsymbol{\delta}(\mathfrak{y}); \tag{3.4}$$

that is to say,

$$\begin{aligned} &\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \wedge y) \mid y \in S\} \qquad (3.5)\\ = \quad &\bigwedge\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to y) \mid y \in S\} \wedge \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\}. \end{aligned}$$

Proof Let z be any formula in S. Then each formula of the form $\mathfrak{y} = \ulcorner y \urcorner \wedge y$ implies the formula $\mathfrak{y} \neq \ulcorner z \urcorner$ if $z \neq y$ and the formula z otherwise; in short, each formula of the form $\mathfrak{y} = \ulcorner y \urcorner \wedge y$ implies each formula of the form $\mathfrak{y} = \ulcorner z \urcorner \to z$. This means that $\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \wedge y) \mid y \in S\}$ implies $\bigwedge\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to y) \mid y \in S\}$. That it implies also $\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\}$ is trivial, and hence the theory named on the left of (3.5) implies the theory named on the right. For the converse, suppose that for each formula y, the formula $\mathfrak{y} = \ulcorner y \urcorner \wedge y$ implies the formula w. Then by modus ponens, $(\mathfrak{y} = \ulcorner y \urcorner \to y) \wedge (\mathfrak{y} = \ulcorner y \urcorner)$ implies w for each y. By conditional proof, $\mathfrak{y} = \ulcorner y \urcorner$ implies $(\mathfrak{y} = \ulcorner y \urcorner \to y) \to w$ for each y. It follows that $\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\}$ implies $(\mathfrak{y} = \ulcorner y \urcorner \to y) \to w$ for each y, and hence that $\bigwedge\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to y) \mid y \in S\} \wedge \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\}$ implies w. The theory named on the right of (3.5) is proved to imply the theory named on the left. ■

If $\mathfrak{y}$ is a structural-descriptive name of a formula y, then one of the elements in the final join $\boldsymbol{\delta}(\mathfrak{y}) = \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\}$ in (3.4) and (3.5) is **L**, and so the entire join disappears: $\boldsymbol{\delta}(\mathfrak{y}) = \mathbf{L}$ and $\boldsymbol{\vartheta}(\mathfrak{y}) = \boldsymbol{\theta}(\mathfrak{y})$. But when $\mathfrak{y}$ demonstrably names no formula, the final term $\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner)\} = \mathbf{S}$.

More interesting than these limiting cases are names of formulas that may loosely be called contingent or factual (though they exist also in incomplete mathematical theories, such as Zermelo set theory $\mathbb{Z}$). An example that is given in Miller (2009) is $\mathfrak{p}$ = 'the first sentence of *Pride & Prejudice*'. Although $\mathfrak{p} \in \mathfrak{N}$ may, for the purposes of this discussion, be assumed to be demonstrable in $\mathbb{T}$, it does not follow that there is any structural-descriptive name $\ulcorner y \urcorner$ for which the formula $\mathfrak{p} = \ulcorner y \urcorner$ belongs to **L**. As observed in (2009), it is because Tarski's scheme (**T**) does not permit the elimination of the word 'true' from such sentences as 'the first sentence of *Pride & Prejudice* is true' that it is not a complete definition of the term 'true sentence'. Such blind ascriptions of truth, however, present no difficulty for the genuine definitions (3.0) and (3.1).

Let us abbreviate the sentence 'It is a truth universally acknowledged, that a single man in possession of a good fortune, must be in want of a wife' by 'p'. It is evident that $\boldsymbol{\vartheta}(\mathfrak{p}) \neq \boldsymbol{\vartheta}(\ulcorner p \urcorner) = \mathbf{Cn}(p) = \boldsymbol{\theta}(\ulcorner p \urcorner) \neq \boldsymbol{\theta}(\mathfrak{p})$, since only a few of the disjuncts in the join $\bigvee\{\mathbf{Cn}(\mathfrak{p} = \ulcorner y \urcorner \wedge y) \mid y \in S\}$ imply the sentence p (one of the successful disjuncts is $\mathbf{Cn}(\mathfrak{p} = \ulcorner p \urcorner \wedge p)$), and equally, only a few of the conjuncts in the meet $\bigwedge\{\mathbf{Cn}(\mathfrak{p} = \ulcorner y \urcorner \to y) \mid y \in S\}$ are implied by p. It will be shown in § 5 that in most cases in which $\mathfrak{y}$ is, in the sense explained, a contingent name of a sentence, each of $\boldsymbol{\vartheta}(\mathfrak{y})$, $\boldsymbol{\theta}(\mathfrak{y})$, and $\boldsymbol{\delta}(\mathfrak{y})$ is an unaxiomatizable theory. In order to prepare for this work, we shall return in the next section to the study of Tarski's general metamathematics.

At this point it is an open question which of the definitions (3.0) and (3.1) is a better definition of truth. As far as material adequacy is concerned, they are on a par (Theorem 1). In favour of the function $\boldsymbol{\theta}$ is the advisability of not making a definition any stronger than necessary. It must be admitted also that infinite meets are easier to work with, and more intuitive, than are infinite joins (as illustrated in the penultimate paragraph of § 2). On the other side, since $=$ (the name of identity sign), for example, is demonstrably the name of no formula, it is satisfactory that $\boldsymbol{\vartheta}(=)$ is the logically false theory $\mathbf{S}$, less satisfactory that $\boldsymbol{\theta}(=)$ is the logically true theory $\mathbf{L}$.

From the point of view of general metamathematics or abstract logic, the definition (3.0) has, however, the decided advantage over (3.1) that it requires, in addition to the elementary logic of identity, the existence among formulas only of the operation $\wedge$ of conjunction, and not the conditional operation $\rightarrow$. It is easily seen, indeed, that even this requirement can be suspended. For the conjunction $x \wedge z$ of two formulas, where it exists, has the same logical force as the meet $\mathbf{x} \wedge \mathbf{z}$ of the axiomatizable systems $\mathbf{x} = \mathbf{Cn}(x)$ and $\mathbf{z} = \mathbf{Cn}(z)$ that x and z axiomatize. The definition (3.0) may therefore be generalized to

$$\boldsymbol{\vartheta}(\mathfrak{y}) \quad = \quad \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \wedge \mathbf{Cn}(y) \mid y \in S\}. \tag{3.6}$$

This definition is widely applicable, for example to many higher-order logics, intensional logics, and paraconsistent logics.

Definition (3.1) is evidently well formed if enough conditionals exist. There is an old result of Skolem's that if the logic $\mathbf{Cn}$ includes the standard rules for the conditional, and disjunction $\vee$ and conjunction $\wedge$ are also present, then the distributive law (2.7) holds. But in the absence of the conditional operation $\rightarrow$ among elements of S, it is not easy, without some concessions to distributivity, to simulate it, or even negation, at the level of deductive theories; it is not easy, that is, to define the conditional of two theories, or the complement of a theory. When $\mathbf{Cn}(x)$ is identical with either $\mathbf{S}$ or $\mathbf{L}$, the theory $\mathbf{Cn}(x \rightarrow z)$ can be readily represented as $\mathbf{Cn}(x)' \vee \mathbf{Cn}(z)$, where $\mathbf{Y}'$ is the authocomplement of the theory $\mathbf{Y}$ (which is defined in (4.3) below), and the identities (4.5) and (4.6) are provable without any assumption of distributivity. The definition

$$\boldsymbol{\tau}(\mathfrak{y}) \quad =_{\mathrm{Df}} \quad \bigwedge\{(\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner))' \vee \mathbf{Cn}(y) \mid y \in S\}, \tag{3.7}$$

is applicable where (3.1) is not, and is still materially adequate: $\boldsymbol{\tau}(\ulcorner x \urcorner) = \mathbf{Cn}(x)$ for all formulas x. But in non-distributive logics its import for contingent names such as $\mathfrak{p}$ is less transparent.

Two features of the definitions (3.0) and (3.1) merit special remark. The first is the scantiness of the resources of Zermelo set theory $\mathbb{Z}$ that are needed in the background theory $\mathbb{T}$ for the formulation of (3.0) and (3.1), whatever may be the logical strength of the system for which truth is being defined. The denumerably many formulas of S, and their structural-descriptive names, can be represented by natural numbers or hereditarily finite sets. S therefore belongs to $\mathcal{V}_{\omega+1}$. Deductive theories are (in general) denumerable sets of formulas, and they too belong to $\mathcal{V}_{\omega+1}$. An ordered pair of elements of $\mathcal{V}_\nu$ belongs to $\mathcal{V}_{\nu+2}$, and therefore the operation $\mathbf{Cn} : \wp(S) \mapsto \wp(S)$, which is a set of ordered pairs of subsets of S, belongs to $\mathcal{V}_{\omega+4}$.

A deductive system $\langle S, \mathbf{Cn}\rangle$ is an ordered pair of an element of $\mathcal{V}_{\omega+1}$ and an element of $\mathcal{V}_{\omega+4}$, and this belongs to $\mathcal{V}_{\omega+6}$. The entire construction fits comfortably into a short extension of the hereditarily finite sets $\mathcal{V}_{\omega}$.

The other most significant feature of the definitions (3.0) and (3.1) is that they are universal. According to many philosophers, one of the chief shortcomings of Tarski's approach is that, although his method of definition can be universalized, there is no universal definition of truth. Each deductive system requires its own work. This disadvantage is plainly overcome here, since both (3.0) and (3.1) contain explicit references to the two components of the variable deductive system $\langle S, \mathbf{Cn}\rangle$. Note also that although the system $\langle S, \mathbf{Cn}\rangle$ for which truth is defined may be distinguished from the system (the background theory $\mathbb{T}$) in which the definition is conducted, there is no need for any hierarchy of distinct metalanguages, syntactical or semantical.

4 Unaxiomatizable Theories

This section is devoted to several characterizations of unaxiomatizable deductive theories. The first characterization (Theorem 4 and its corollaries) is well known, and is effective for every consequence operation $\mathbf{Cn}$. On the way to the final characterization (Theorem 18), which requires that $\mathbf{Cn}$ include the whole of classical logic, is a result (Theorem 10) for distributive consequence operations that relates the axiomatizability of some theories with that of others.

4.1 Sequences of Theories

An infinite sequence $\{\mathbf{y}_j \mid j \in \mathcal{N}\}$ of axiomatizable theories is (for reasons outlined in § 2 above) called *decreasing* if $\mathbf{y}_i \subseteq \mathbf{y}_k$ whenever $i < k$, and *strictly decreasing* if in addition $\mathbf{y}_k \not\subseteq \mathbf{y}_i$ whenever $i < k$. An infinite sequence $\{y_j \mid j \in \mathcal{N}\}$ of formulas is likewise called *decreasing* if $y_k \vdash y_i$ whenever $i < k$, and *strictly decreasing* if also $y_i \nvdash y_k$ whenever $i < k$.

Theorem 4 The theory $\mathbf{Y}$ is unaxiomatizable if & only if $\mathbf{Y} = \mathbf{Cn}(\bigcup\{\mathbf{y}_j \mid j \in \mathcal{N}\})$ for some strictly decreasing sequence $\{\mathbf{y}_j \mid j \in \mathcal{N}\}$ of axiomatizable theories.

Proof Suppose that $\mathbf{Y} = \mathbf{Cn}(\bigcup\{\mathbf{y}_j \mid j \in \mathcal{N}\})$ is axiomatized by the finite set $\{y_0, \ldots, y_{k-1}\}$. Then by (2.1) there is for each $l < k$ some finite i_l for which $\mathbf{Cn}(\mathbf{y}_{i_l}) \vdash y_l$. If i is the maximum of these i_l then, since the sequence $\{\mathbf{y}_j \mid j \in \mathcal{N}\}$ is strictly decreasing, $\mathbf{y}_i \vdash y_l$ for each $l < k$, and so $\mathbf{Y} \subseteq \mathbf{y}_i$. But $\mathbf{y}_{i+1} \subseteq \mathbf{Y}$, and hence $\mathbf{y}_{i+1} \subseteq \mathbf{y}_i$, contrary to assumption. To represent as $\mathbf{Cn}(\bigcup\{\mathbf{y}_j \mid j \in \mathcal{N}\})$ any given unaxiomatizable theory $\mathbf{Y}$, enumerate its consequences $\{y_i \mid i \in \mathcal{N}\}$, and then define $\mathbf{y}_0$ as $\mathbf{Cn}(\{y_0\})$ and $\mathbf{y}_{j+1}$ as $\mathbf{Cn}(\{y_0, \ldots, y_k\})$ where k is the least number for which $\mathbf{Cn}(\{y_0, \ldots, y_k\}) \not\subseteq \mathbf{y}_j$. It is clear that the theories $\mathbf{y}_j$ form a strictly decreasing sequence. ■

Corollary 5 (A. Robinson) If the logic $\mathbf{Cn}$ includes conjunction, the theory $\mathbf{Y}$ is unaxiomatizable if & only if $\mathbf{Y} = \mathbf{Cn}(\{y_j \mid j \in \mathcal{N}\})$ for some strictly decreasing sequence $\{y_j \mid j \in \mathcal{N}\}$.

Proof The proof is essentially the same as the proof of the theorem. ■

Corollary 6 In elementary logic with identity, let ω_j say that the universe has cardinality j. The maximal theory $\bigwedge\{\mathbf{Cn}(\neg\omega_j) \mid j > 0\}$ identified at the end of § 2 is unaxiomatizable.

Proof Define the sentence y_j as the conjunction $\neg\omega_1 \wedge \cdots \wedge \neg\omega_j$, so that $y_0 = \top$. The import of y_j is that the universe has cardinality greater than j. The sequence $\{y_j \mid j \in \mathcal{N}\}$ is obviously strictly decreasing. ∎

Corollary 7 Let $\{y_j \mid j \in \mathcal{N}\}$ be any enumeration of a set Y of formulas, $X_i = \{y_j \mid j < i\}$, and $\mathbf{x}_i = \mathbf{Cn}(X_i)$ for each $i \in \mathcal{N}$. Then $\mathbf{Cn}(Y)$ is axiomatizable if & only if the decreasing sequence $\{\mathbf{x}_i \mid i \in \mathcal{N}\}$ of axiomatizable theories is eventually constant.

Proof $\mathbf{Cn}(Y)$ is axiomatizable if & only if $\mathbf{Cn}(Y) = \mathbf{x}_i$ for some $i \in \mathcal{N}$. Since the sequence $\{\mathbf{x}_i \mid i \in \mathcal{N}\}$ is decreasing, and each $\mathbf{x}_k \subseteq \mathbf{Cn}(Y)$, this holds if & only if $\mathbf{x}_i = \mathbf{x}_k$ for all $k \geq i$. ∎

4.2 A Consequence of Modularity

The least intuitive of the results that concern disjunctions and conjunctions only, holding whenever the logic **Cn** includes the distributive (in fact, the modular) law, is Theorem 18 of Tarski (1935–1936). It leads to an alternative criterion of unaxiomatizability in classical logic, in terms of failure of the law of non-contradiction. Since there appear not to exist published proofs of most of these results, we shall prove them here. Three preparatory lemmas are required. Corollary 9 and Lemma 10 are included in Lemmas 1 and 2 on pp. 113f. of Birkhoff (1973).

Lemma 8 Suppose that $x \wedge z$ is always defined. If $\mathbf{X} \vee \mathbf{Z}$ is axiomatizable, there are axiomatizable theories $\mathbf{x} \subseteq \mathbf{X}$ and $\mathbf{z} \subseteq \mathbf{Z}$ such that $\mathbf{X} \vee \mathbf{Z} = \mathbf{x} \vee \mathbf{z}$.

Proof If $\mathbf{X} \vee \mathbf{Z} = \varnothing$ then we may identify both $\mathbf{x}$ and $\mathbf{z}$ with $\varnothing$. If $\mathbf{X} \vee \mathbf{Z} \neq \varnothing$, the existence of conjunctions implies that $\mathbf{X} \vee \mathbf{Z} = \mathbf{Cn}(y)$ for some formula $y \in \mathbf{X} \vee \mathbf{Z}$, from which it follows that $\mathbf{X} \vee \mathbf{Z} = \mathbf{y} \vee \mathbf{y}$, where $\mathbf{y} = \mathbf{Cn}(y)$ is an axiomatizable subtheory of both $\mathbf{X}$ and $\mathbf{Z}$. ∎

Corollary 9 (Birkhoff) Suppose that $x \vee z$ is always defined. Then

$$\mathbf{X} \vee \mathbf{Z} \quad = \quad \{x \vee z \mid \mathbf{X} \vdash x \text{ and } \mathbf{Z} \vdash z\}, \tag{4.0}$$

Proof If $\mathbf{X} \vdash x$ and $\mathbf{Z} \vdash z$ then both $\mathbf{X}$ and $\mathbf{Z}$ imply $x \vee z$, and so $\mathbf{X} \vee \mathbf{Z} \vdash x \vee z$. Since y is equivalent to $y \vee y$, every formula in $\mathbf{X} \vee \mathbf{Z}$ is a disjunction of formulas from $\mathbf{X}$ and from $\mathbf{Z}$. ∎

Lemma 10 (Tarski) If $\mathbf{X} \wedge \mathbf{Z}$ is axiomatizable, then there are axiomatizable theories $\mathbf{x} \subseteq \mathbf{X}$ and $\mathbf{z} \subseteq \mathbf{Z}$ such that $\mathbf{X} \wedge \mathbf{Z} = \mathbf{x} \wedge \mathbf{z}$.

Proof If $\mathbf{X} \wedge \mathbf{Z} = \mathbf{Cn}(\{u_0, \ldots u_{j-1}\})$, then by (2.1) there are axiomatizable theories $\mathbf{y}_0, \ldots \mathbf{y}_{j-1}$, subsets of $\mathbf{X}$, and axiomatizable theories $\mathbf{w}_0, \ldots \mathbf{w}_{j-1}$, subsets of $\mathbf{Z}$, such that $u_l \in \mathbf{Cn}(\mathbf{y}_l \cup \mathbf{w}_l) = \mathbf{y}_l \wedge \mathbf{w}_l$ for each $l < j$. Let $\mathbf{y} = \mathbf{y}_0 \wedge \ldots \wedge \mathbf{y}_{j-1}$ and $\mathbf{w} = \mathbf{w}_0 \wedge \ldots \wedge \mathbf{w}_{j-1}$. It is immediate that $\mathbf{y} \subseteq \mathbf{X}$ and $\mathbf{w} \subseteq \mathbf{Z}$ are axiomatizable, and that $\mathbf{Cn}(\{u_0, \ldots u_{j-1}\}) \subseteq \mathbf{y} \wedge \mathbf{w} \subseteq \mathbf{X} \wedge \mathbf{Z} = \mathbf{Cn}(\{u_0, \ldots u_{j-1}\})$. We may conclude that $\mathbf{X} \wedge \mathbf{Z} = \mathbf{y} \wedge \mathbf{w}$. ∎

LEMMA 11 Suppose that $x \wedge z$ is always defined. If $\mathbf{X} \vee \mathbf{Z}$ and $\mathbf{X} \wedge \mathbf{Z}$ are axiomatizable, then there are axiomatizable theories $\mathbf{x} \subseteq \mathbf{X}$ and $\mathbf{z} \subseteq \mathbf{Z}$ such that $\mathbf{X} \vee \mathbf{Z} = \mathbf{x} \vee \mathbf{z}$ and $\mathbf{X} \wedge \mathbf{Z} = \mathbf{x} \wedge \mathbf{z}$.

PROOF By Lemmas 8 and 10, there are axiomatizable theories $\mathbf{u}, \mathbf{y} \subseteq \mathbf{X}$ and axiomatizable theories $\mathbf{v}, \mathbf{w} \subseteq \mathbf{Z}$ for which $\mathbf{X} \vee \mathbf{Z} = \mathbf{u} \vee \mathbf{v}$ and $\mathbf{X} \wedge \mathbf{Z} = \mathbf{y} \wedge \mathbf{w}$. Set $\mathbf{x} = \mathbf{u} \wedge \mathbf{y}$ and $\mathbf{z} = \mathbf{v} \wedge \mathbf{w}$. Since $\mathbf{u}$ and $\mathbf{y}$ are subsets of $\mathbf{X}$, so is $\mathbf{x}$, and likewise $\mathbf{z} \subseteq \mathbf{Z}$. This implies that $\mathbf{x} \vee \mathbf{z} \subseteq \mathbf{X} \vee \mathbf{Z}$. It is clear too that $\mathbf{u} \subseteq \mathbf{x}$ and $\mathbf{v} \subseteq \mathbf{z}$, so that $\mathbf{X} \vee \mathbf{Z} = \mathbf{u} \vee \mathbf{v} \subseteq \mathbf{x} \vee \mathbf{z}$. We may conclude that $\mathbf{X} \vee \mathbf{Z} = \mathbf{x} \vee \mathbf{z}$. In the same way, $\mathbf{x} \wedge \mathbf{z} \subseteq \mathbf{X} \wedge \mathbf{Z} = \mathbf{y} \wedge \mathbf{w} \subseteq (\mathbf{u} \wedge \mathbf{y}) \wedge (\mathbf{v} \wedge \mathbf{w}) = \mathbf{x} \wedge \mathbf{z}$. We may conclude that $\mathbf{X} \wedge \mathbf{Z} = \mathbf{x} \wedge \mathbf{z}$. ■

THEOREM 12 (Tarski) Suppose that the distributive law (2.7) holds in the logic $\mathbf{Cn}$. Then if $\mathbf{X} \vee \mathbf{Z}$ and $\mathbf{X} \wedge \mathbf{Z}$ are both axiomatizable, $\mathbf{X}$ and $\mathbf{Z}$ are both axiomatizable.

PROOF By Lemma 11 there are axiomatizable theories $\mathbf{x} \subseteq \mathbf{X}$ and $\mathbf{z} \subseteq \mathbf{Z}$ for which $\mathbf{X} \vee \mathbf{Z} = \mathbf{x} \vee \mathbf{z}$ and $\mathbf{X} \wedge \mathbf{Z} = \mathbf{x} \wedge \mathbf{z}$. It is plain that $\mathbf{x} \vee \mathbf{Z} \subseteq \mathbf{X} \vee \mathbf{Z}$; and moreover, if $\mathbf{X} \vee \mathbf{Z} \vdash y$, then $\mathbf{x} \vee \mathbf{z} \vdash y$, and so $\mathbf{x} \vee \mathbf{Z} \vdash y$. It follows that $\mathbf{X} \vee \mathbf{Z} = \mathbf{x} \vee \mathbf{Z}$. In the same way, $\mathbf{X} \wedge \mathbf{Z} = \mathbf{x} \wedge \mathbf{Z}$. Using (2.9), which is a consequence of (2.7), and the absorption laws (2.8), we continue

$$\begin{aligned} \mathbf{X} = \mathbf{X} \vee (\mathbf{X} \wedge \mathbf{Z}) &= \mathbf{X} \vee (\mathbf{x} \wedge \mathbf{Z}) \quad (4.1) \\ &= [\mathbf{X} \vee \mathbf{x}] \wedge [\mathbf{X} \vee \mathbf{Z}] \\ &= \mathbf{x} \wedge (\mathbf{x} \vee \mathbf{Z}) \\ &= \mathbf{x}. \end{aligned}$$

$\mathbf{Z} = \mathbf{z}$ is proved likewise. It should be noted that, since $\mathbf{x} \subseteq \mathbf{X}$, at line (4.1) the modular law

$$\mathbf{Y} \vdash \mathbf{X} \quad \Leftrightarrow \quad \mathbf{Y} \vee (\mathbf{X} \wedge \mathbf{Z}) = (\mathbf{Y} \vee \mathbf{X}) \wedge (\mathbf{Y} \vee \mathbf{Z}) \qquad (4.2)$$

(for deductive theories) suffices in place of the full distributive law (2.9). ■

COROLLARY 13 If the modular law (4.2) holds for the logic $\mathbf{Cn}$, then $\mathbf{X} \vee \mathbf{Z}$ and $\mathbf{X} \wedge \mathbf{Z}$ are both axiomatizable if & only if $\mathbf{X}$ and $\mathbf{Z}$ are both axiomatizable.

PROOF The modular law is not needed for the converse of Theorem 12. Suppose that $\mathbf{X}$ and $\mathbf{Z}$ are axiomatizable. It is evident that no conditions on $\mathbf{Cn}$ are needed to guarantee that $\mathbf{X} \wedge \mathbf{Z}$ is axiomatizable. The existence of both disjunction and conjunction is plainly sufficient (but not necessary) for $\mathbf{X} \vee \mathbf{Z}$ to be axiomatizable. In the absence of conjunction, however, there may be no axiomatization of $\mathbf{X} \vee \mathbf{Z}$ when $\mathbf{X} = \mathbf{Cn}\{x_0, x_1\}$ and $\mathbf{Z} = \mathbf{Cn}(z)$; and in the absence of disjunction, there may be no axiomatization of $\mathbf{X} \vee \mathbf{Z}$ even if $\mathbf{X} = \mathbf{Cn}(x)$ and $\mathbf{Z} = \mathbf{Cn}(z)$. ■

4.3 Authocomplementation

When we come to define the complement $\mathbf{Y}'$ of an unaxiomatizable theory $\mathbf{Y}$ we inevitably enter non-classical territory. For by Theorem 12, if $\mathbf{Y}$ is unaxiomatizable then at least one of $\mathbf{Y} \vee \mathbf{Y}'$ and $\mathbf{Y} \wedge \mathbf{Y}'$ is unaxiomatizable, no matter which theory we decide to identify Y' with. Since both L and S are axiomatizable in classical logic, it follows that either the law of excluded middle $\mathsf{Y} \vee \mathsf{Y}' = \mathsf{L}$ or the law of non-contradiction $\mathsf{Y} \wedge \mathsf{Y}' = \mathsf{S}$ must fail for each unaxiomatizable theory Y. In fact, it is always the latter, classically and elsewhere. Following Tarski (1935–1936), Theorem 12(a), we therefore define what we shall call the *authocomplement*

$$\mathsf{Y}' \quad = \quad \bigwedge\{\mathsf{Z} \mid \vdash \mathsf{Y} \vee \mathsf{Z}\} \tag{4.3}$$

of the theory Y. If we assume the infinite distributive law (2.10), or the more basic law (2.7), we may derive from (4.3) some simple consequences (including the law of excluded middle (4.4), one law of contraposition (4.8), one De Morgan law (4.9), and the law of triple negation (4.11)).

THEOREM 14 If the consequence operation Cn obeys the distributive law (2.7), then the following identities, implications, and equivalences hold for all theories X, Y, Z, and families $\mathcal{K}$.

$$\mathsf{Y} \vee \mathsf{Y}' \quad = \quad \mathsf{L} \tag{4.4}$$

$$\mathsf{S}' \quad = \quad \mathsf{L} \tag{4.5}$$

$$\mathsf{L}' \quad = \quad \mathsf{S} \tag{4.6}$$

$$\mathsf{X}' \vdash \mathsf{Z} \quad \Leftrightarrow \quad \vdash \mathsf{X} \vee \mathsf{Z} \tag{4.7}$$

$$\mathsf{X}' \vdash \mathsf{Z} \quad \Leftrightarrow \quad \mathsf{Z}' \vdash \mathsf{X} \tag{4.8}$$

$$(\bigwedge \mathcal{K})' \quad = \quad \bigvee\{\mathsf{Y}' \mid \mathsf{Y} \in \mathcal{K}\} \tag{4.9}$$

$$\mathsf{Y}'' \quad \vdash \quad \mathsf{Y} \tag{4.10}$$

$$\mathsf{Y}''' \quad = \quad \mathsf{Y}'. \tag{4.11}$$

PROOF These results are all straightforward consequences of (4.3) and (2.7). The proof of (4.4) is much facilitated by resort to (2.10). ■

COROLLARY 15 Suppose that the logic Cn includes the distributive law (2.7), and that S is axiomatizable. If $\mathsf{Y} \wedge \mathsf{Y}' = \mathsf{S}$, then Y is axiomatizable.

PROOF Since $\mathsf{L} = \mathsf{Cn}(\varnothing)$ is always axiomatizable, it follows from (4.4) and Theorem 12 that both Y and Y' are axiomatizable. Since it is an immediate consequence of (4.5) that $\mathsf{S} \wedge \mathsf{S}' = \mathsf{S}$, the condition that S be axiomatizable is essential if the corollary is to hold. ■

COROLLARY 16 Suppose that the logic Cn obeys the distributive law (2.7), and that S is axiomatizable. If $\boldsymbol{\Omega}$ is a theory that is both maximal and unaxiomatizable, then $\boldsymbol{\Omega}' = \mathsf{L}$.

PROOF By the previous corollary, $\boldsymbol{\Omega} \wedge \boldsymbol{\Omega}' \neq \mathsf{S}$. By the definition of maximality, $\boldsymbol{\Omega} \vdash \boldsymbol{\Omega}'$. This means that $\boldsymbol{\Omega}' = \boldsymbol{\Omega} \vee \boldsymbol{\Omega}'$, which is L by (4.4). ■

As in the case of disjunction noted in the proof of Corollary 13, the absence of the corresponding connective in the system $\langle S, \mathsf{Cn}\rangle$ may render compounds of axiomatizable theories unaxiomatizable. If S is infinite then it may not be axiomatizable (for example, if $\mathsf{Cn}(Y) = Y$ for every $Y \subseteq S$), even though by (4.5) it is the complement of an axiomatizable theory L; and if the logic Cn does not support a properly behaved negation $\neg y$ of each formula y, then $\mathbf{Y}'$ may be unaxiomatizable even if $\mathbf{Y} = \mathbf{Cn}(y)$. As might have been expected, if $\neg$ is classical, all is well.

LEMMA 17 If the logic $\mathbf{Cn}$ includes the classical negation operation $\neg$, then $\mathbf{Cn}(y)' = \mathbf{Cn}(\neg y)$.

PROOF Classical logic guarantees that $y \wedge \neg y \vdash \mathbf{Z}$ for any y and any $\mathbf{Z}$. Now if $\vdash \mathbf{Cn}(y) \vee \mathbf{Z}$ then $\vdash y \vee \mathbf{Z}$, and hence also $\neg y \vdash \mathbf{Z}$, and hence $\mathbf{Cn}(\neg y) \vdash \mathbf{Z}$. It follows that $\mathbf{Cn}(\neg y)$ implies the meet $\bigwedge\{\mathbf{Z} \mid \vdash \mathbf{Cn}(y) \vee \mathbf{Z}\} = \mathbf{Cn}(y)'$. Classical logic guarantees too that $\vdash y \vee \neg y$ for any y. That is, $\vdash \mathbf{Cn}(y) \vee \mathbf{Cn}(\neg y)$, from which we may conclude by (4.7) that $\mathbf{Cn}(y)' \vdash \mathbf{Cn}(\neg y)$. ■

THEOREM 18 If $\mathbf{Cn}$ is classical, then $\mathbf{Y}$ is an axiomatizable theory if & only if $\mathbf{Y} \wedge \mathbf{Y}' = \mathbf{S}$.

PROOF To establish the converse to Corollary 15, it is necessary to note only that if $\mathbf{Y} = \mathbf{Cn}(y)$ then $\mathbf{Y}' = \mathbf{Cn}(\neg y)$, and hence $\mathbf{Y} \wedge \mathbf{Y}' = \mathbf{Cn}(y) \wedge \mathbf{Cn}(\neg y) = \mathbf{Cn}(y \wedge \neg y) = \mathbf{S}$. ■

To sum up: whereas in intuitionistic logic the negation $\neg y$ of a formula y contradicts y, but does not always complement it (the law of excluded middle sometimes fails), in the calculus of theories based on distributive consequence operations the theory $\mathbf{Y}'$ is an authentic complement of $\mathbf{Y}$, but does not always contradict it (the equation $\mathbf{Y} \wedge \mathbf{Y}' = \mathbf{S}$ sometimes fails). Since the algebraic counterpart of intuitionistic negation is usually called *pseudocomplementation*, it is appropriate to call the operation $'$ on theories based on a distributive consequence operation $\mathbf{Cn}$ an operation of *authocomplementation*. The conditional (relative pseudocomplement) $x \to z$ of intuitionistic logic likewise does not survive unscathed between theories, though it is always possible to define a conditional $\mathbf{X} \twoheadrightarrow \mathbf{Z}$ if the antecedent theory $\mathbf{X}$ is finitely axiomatizable.

LEMMA 19 If $\mathbf{Cn}$ is classical and $\mathbf{X} = \mathbf{Cn}(x)$, then the theory $\mathbf{Y} = \mathbf{Cn}(\neg x) \vee \mathbf{Z}$ complies with both the laws of *modus ponens*: $\mathbf{X} \wedge \mathbf{Y} \vdash \mathbf{Z}$, and *conditional proof*: if $\mathbf{X} \wedge \mathbf{W} \vdash \mathbf{Z}$ then $\mathbf{W} \vdash \mathbf{Y}$.

PROOF For modus ponens use Lemma 17 and Theorem 18. For conditional proof, note that if $x \wedge \mathbf{W}$ implies each $z \in \mathbf{Z}$, then $\mathbf{W}$ implies each $x \to z$, that is, each $\neg x \vee z$. But the formulas of the form $\neg x \vee z$ (where $z \in \mathbf{Z}$) are exactly the consequences of $\mathbf{Cn}(\neg x) \vee \mathbf{Z}$, which is $\mathbf{Y}$. ■

Other conditionals sometimes exist. The theory $\boldsymbol{\delta}(\mathfrak{y})$ defined in (3.2), for example, is the conditional $\boldsymbol{\theta}(\mathfrak{y}) \twoheadrightarrow \boldsymbol{\vartheta}(\mathfrak{y})$ (and symmetrically, $\boldsymbol{\theta}(\mathfrak{y}) = \boldsymbol{\delta}(\mathfrak{y}) \twoheadrightarrow \boldsymbol{\vartheta}(\mathfrak{y})$). But if $\boldsymbol{\Omega}$ is an unaxiomatizable maximal theory that does not imply z, then there is no conditional $\boldsymbol{\Omega} \twoheadrightarrow \mathbf{z}$. For $\boldsymbol{\Omega} \vdash x \to z$ if & only if $\boldsymbol{\Omega} \vdash \neg x$, and (since $\boldsymbol{\Omega}$ is unaxiomatizable) there is no logically weakest such x.

The *remainder* or relative authocomplement $\mathbf{Z} - \mathbf{X}$ may be defined for any two theories by a generalization of (4.3): $\mathbf{X} - \mathbf{Y} =_{\mathrm{Df}} \bigwedge\{\mathbf{Z} \mid \mathbf{X} \vdash \mathbf{Y} \vee \mathbf{Z}\}$. But it will not be needed in this paper.

5 The Unaxiomatizability of Truth

The present section establishes that if $\mathfrak{y}$ is replaced by a variable name of a formula, the theories $\boldsymbol{\vartheta}(\mathfrak{y})$ and $\boldsymbol{\theta}(\mathfrak{y})$ are unaxiomatizable when **Cn** is classical logic; and that, provided that **Cn** is recursively presented and rich enough to include Peano arithmetic, $\boldsymbol{\delta}(\mathfrak{y})$ is unaxiomatizable. It must be remembered that we are silently assuming that the logic **Cn** includes the whole of the standard elementary logic of identity, and also enough apparatus to allow construction of, and effective manipulations with, structural-descriptive names. The three proofs to be given differ, and only the results concerning $\boldsymbol{\delta}(\mathfrak{y})$ require that, in addition, **Cn** be stronger than elementary classical logic.

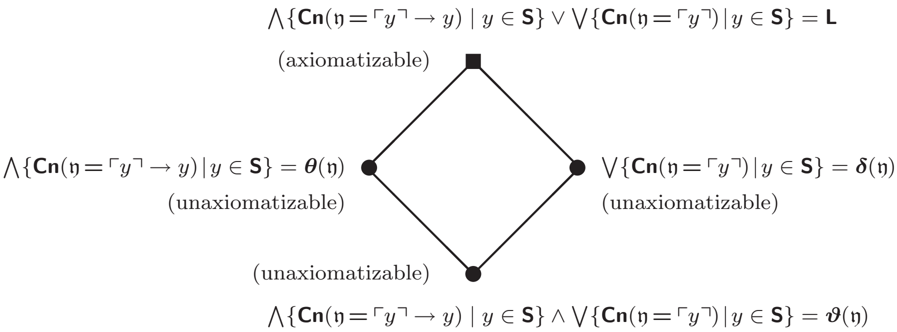

Figure 0: The theories $\boldsymbol{\vartheta}(\mathfrak{y})$, $\boldsymbol{\theta}(\mathfrak{y})$, $\boldsymbol{\delta}(\mathfrak{y})$, and **L**

We show first that if the logic **Cn** contains enough of classical logic, then $\boldsymbol{\theta}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y}) = \mathsf{L}$ (Theorem 20). In these circumstances $\boldsymbol{\delta}(\mathfrak{y}) = \boldsymbol{\theta}(\mathfrak{y}) \twoheadrightarrow \boldsymbol{\vartheta}(\mathfrak{y})$, and the four theories $\boldsymbol{\vartheta}(\mathfrak{y})$, $\boldsymbol{\theta}(\mathfrak{y})$, $\boldsymbol{\delta}(\mathfrak{y})$, and **L** form a lattice quadrilateral, as shown in Figure 0. We then use Theorem 4 to prove that if **Cn** is non-trivial and classical and contains the rule of existential introduction, the theory $\boldsymbol{\theta}(\mathfrak{y})$ is unaxiomatizable (Theorem 24). An application of Theorem 3 and Theorem 12 then shows that, provided **Cn** is distributive, the theory $\boldsymbol{\vartheta}(\mathfrak{y})$ is also unaxiomatizable (Theorem 25). These are the main results. More work, and more assumptions, are needed for the results, admittedly less central, concerning the unaxiomatizability of the theory $\boldsymbol{\delta}(\mathfrak{y})$ (Lemma 28, Theorem 30).

Theorem 20 In every distributive logic **Cn** in which $(x \to z) \vee x$ is a theorem, $\boldsymbol{\theta}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y}) = \mathsf{L}$.

Proof The join $\boldsymbol{\theta}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y}) \vdash w$ if & only if both $\boldsymbol{\theta}(\mathfrak{y}) \vdash w$ and $\boldsymbol{\delta}(\mathfrak{y}) \vdash w$. Thanks to finitariness (2.1), the former implies that there exists some finite set of formulas $\{y_i \mid i < k\} \subseteq S$ such that $(\mathfrak{y}=\ulcorner y_0\urcorner \to y_0) \wedge \cdots \wedge (\mathfrak{y}=\ulcorner y_{k-1}\urcorner \to y_{k-1}) \vdash w$, and so $\bigwedge\{\mathfrak{y}=\ulcorner y_i\urcorner \to y_i \mid i < k\} \vee \boldsymbol{\delta}(\mathfrak{y}) \vdash w$. It follows by (2.10) that $\bigwedge\{(\mathfrak{y}=\ulcorner y_i\urcorner \to y_i) \vee \boldsymbol{\delta}(\mathfrak{y}) \mid i < k\} \vdash w$, and hence that the set of formulas $\{(\mathfrak{y}=\ulcorner y_i\urcorner \to y_i) \vee (\mathfrak{y}=\ulcorner y_i\urcorner) \mid i < k\} \vdash w$. The assumption that $(x \to z) \vee x$ is a theorem of the logic **Cn** now allows us to conclude that $w \in \mathsf{L}$, and in consequence that $\boldsymbol{\theta}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y}) = \mathsf{L}$. ■

Corollary 21 In every distributive logic $\mathbf{Cn}$, $\boldsymbol{\tau}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y}) = \mathbf{L}$.

Proof According to its definition (3.7), $\boldsymbol{\tau}(\mathfrak{y}) = \bigwedge\{(\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner))' \vee \mathbf{Cn}(y) \mid y \in S\}$, where $'$ is the authocomplementation operation on theories defined in (4.3); and so by (2.10) and (3.2), $\boldsymbol{\tau}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y}) = \bigwedge\{(\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner))' \vee \mathbf{Cn}(y) \vee \boldsymbol{\delta}(\mathfrak{y}) \mid y \in S\}$. This implies that if $\boldsymbol{\tau}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y}) \vdash w$ then

$$\bigwedge\{(\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner))' \vee \mathbf{Cn}(y) \vee \mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\} \vdash w.$$

But by (4.4), $(\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner))' \vee \mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) = \mathbf{L}$, and so $w \in \mathbf{L}$, and in consequence $\boldsymbol{\tau}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y}) = \mathbf{L}$. ∎

Lemma 22 Suppose that $\mathbf{Cn}$ contains the classical rules for the conditional $\to$ and for existential introduction. If y does not contain x free then $x = \ulcorner y \urcorner \to y$ is a theorem (that is, belongs to $\mathbf{L}$) if & only if y is a theorem.

Proof If the formula $x = \ulcorner y \urcorner \to y \in \mathbf{L}$, then $x = \ulcorner y \urcorner \vdash y$, and accordingly $\exists x(x = \ulcorner y \urcorner) \vdash y$. That is to say, if $x = \ulcorner y \urcorner \to y$ is a theorem then so is y. The converse is immediate. ∎

Lemma 23 If $\mathbf{Cn}$ contains the classical rules for the conditional $\to$, and the rule of *ex falso quodlibet* (from z and $\neg z$ every formula may be derived), and Y is a subset of $\{\mathfrak{y} = \ulcorner y \urcorner \to y) \mid y \in S\}$ that contains no theorems, then no proper subset of Y implies any other element of Y.

Proof Thanks to (2.1) we need consider only a finite subset $X = \{\mathfrak{y} = \ulcorner y_i \urcorner \to y_i \mid i < k\} \subseteq Y$, and an element $\mathfrak{y} = \ulcorner y_k \urcorner \to y_k \in Y \setminus X$. If $\bigwedge\{\mathfrak{y} = \ulcorner y_i \urcorner \to y_i \mid i < k\} \vdash \mathfrak{y} = \ulcorner y_k \urcorner \to y_k$, then $\bigwedge\{\mathfrak{y} \neq \ulcorner y_i \urcorner \mid i < k\} \vdash \mathfrak{y} = \ulcorner y_k \urcorner \to y_k$. But $\mathfrak{y} = \ulcorner y_k \urcorner \vdash \mathfrak{y} \neq \ulcorner y_i \urcorner$ for each $i < k$ (since the structural-descriptive names name distinct formulas), and hence $\mathfrak{y} = \ulcorner y_k \urcorner \vdash \mathfrak{y} = \ulcorner y_k \urcorner \to y_k$, and hence $\vdash \mathfrak{y} = \ulcorner y_k \urcorner \to y_k$. This is an impossible conclusion if Y contains no theorems. ∎

Theorem 24 If $\mathbf{Cn}$ is non-trivial and distributive, and contains the rules assumed in Lemma 22 and Lemma 23, then the theory $\boldsymbol{\theta}(\mathfrak{y}) = \bigwedge\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to y) \mid y \in S\}$ is unaxiomatizable.

Proof Since $\mathbf{Cn}$ is non-trivial, there are infinitely many formulas y that are not theorems. By Lemma 22 there are infinitely many y such that the formula $x = \ulcorner y \urcorner \to y$ is not a theorem, and hence the subset Y of $\{\mathfrak{y} = \ulcorner y \urcorner \to y) \mid y \in S\}$ that contains no theorems is infinite; and since these two sets differ only in the theorems they contain, $\mathbf{Cn}(Y) = \boldsymbol{\theta}(\mathfrak{y})$. Since S is countable, the elements of Y can be listed as $\{y_i \mid i < k\}$, which is equivalent to the sequence $\{z_i \mid i < k\}$ defined by $z_0 = y_0$ and $z_{k+1} = z_k \wedge y_{k+1}$ for every $k \in \mathcal{N}$. By Lemma 23, this sequence is strictly decreasing. By Theorem 4, $\boldsymbol{\theta}(\mathfrak{y}) = \mathbf{Cn}(Y)$ is unaxiomatizable. ∎

Theorem 25 If $\mathbf{Cn}$ is non-trivial and distributive, and contains the rules assumed in Lemma 22 and Lemma 23, then the theory $\boldsymbol{\vartheta}(\mathfrak{y})$ is unaxiomatizable.

Proof Since $\mathbf{L} = \boldsymbol{\theta}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y})$ is axiomatizable in all logics and, by Theorem 24, $\boldsymbol{\theta}(\mathfrak{y})$ is unaxiomatizable in all sufficiently strong non-trivial logics, we may apply Theorem 12 to conclude that $\boldsymbol{\vartheta}(\mathfrak{y}) = \boldsymbol{\theta}(\mathfrak{y}) \wedge \boldsymbol{\delta}(\mathfrak{y})$ is unaxiomatizable in all these subsystems of classical logic. ∎

At this stage nothing has been proved about the theory $\boldsymbol{\delta}(\mathfrak{y})$. For the remainder of this section it will be assumed that **Cn** contains the whole of classical logic. We shall show in Corollary 27 that in an ω-complete logic **Cn** (that is to say, one in which the set **L** is ω-complete), $\boldsymbol{\delta}(\mathfrak{y}) = \mathbf{L}$, and accordingly $\boldsymbol{\vartheta}(\mathfrak{y}) = \boldsymbol{\theta}(\mathfrak{y})$. But in some richer logics, including those that are ω-consistent as well as essentially incomplete, $\boldsymbol{\delta}(\mathfrak{y})$ is unaxiomatizable. The properties of ω-completeness and ω-consistency are discussed by Tarski (1933b), and also in many modern textbooks.

A formula $z(\mathfrak{y})$ in S will be called $\mathfrak{y}$-*universal* if each instance $z(\ulcorner y \urcorner)$ obtained by substituting for $\mathfrak{y}$ a structural-descriptive name $\ulcorner y \urcorner$ of a formula y in S is a theorem in **Cn**. If the variable $\mathfrak{y}$ is not free in z, then z is $\mathfrak{y}$-*universal* if & only if it is a theorem in **Cn**, an element of **L**.

LEMMA 26 The theory $\boldsymbol{\delta}(\mathfrak{y})$ is identical with the meet of all the $\mathfrak{y}$-universal formulas of S: $\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\} = \bigwedge\{z(\mathfrak{y}) \mid z(\mathfrak{y}) \in S \;\&\; z \text{ is } \mathfrak{y}\text{-universal}\}$.

PROOF Let $z(\mathfrak{y})$ be a $\mathfrak{y}$-universal formula. It is plain that $\mathfrak{y} = \ulcorner y \urcorner \vdash z(\mathfrak{y})$ for any $y \in S$, and hence that $\boldsymbol{\delta}(\mathfrak{y}) = \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\} \vdash z(\mathfrak{y})$. For the converse, suppose that $z(\mathfrak{y})$ is derivable from each formula of the form $\mathfrak{y} = \ulcorner y \urcorner$. It is immediate that each $z(\ulcorner y \urcorner)$ is derivable from $\ulcorner y \urcorner = \ulcorner y \urcorner$, and accordingly is a theorem. That is to say, z is a $\mathfrak{y}$-universal formula. ■

COROLLARY 27 If the logic **Cn** is ω-complete, then $\boldsymbol{\delta}(\mathfrak{y}) = \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in \mathbf{S}\} = \mathbf{L}$.

PROOF This is immediate, using Lemma 26. If $z(\mathfrak{y})$ is $\mathfrak{y}$-universal in an ω-complete logic, then $\forall \mathfrak{y} z(\mathfrak{y})$ is a theorem. It follows that $\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in \mathbf{S}\} = \mathbf{L}$. ■

THEOREM 28 If the logic **Cn** is ω-incomplete, then $\boldsymbol{\delta}(\mathfrak{y}) = \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in \mathbf{S}\} \neq \mathbf{L}$.

PROOF This is also practically immediate, using Lemma 26. If $\bigwedge\{z(\mathfrak{y}) \mid z(\mathfrak{y}) \in S \;\&\; z \text{ is } \mathfrak{y}\text{-universal}\} = \mathbf{L}$, then every $\mathfrak{y}$-universal formula $z(\mathfrak{y})$ is a theorem, which implies that **Cn** is ω-complete. ■

For an example, let g be an effective $1-1$ association in $\mathbb{ZF}$ of sentences with proofs, and let u be an undecidable sentence in $\mathbb{ZF}$ that is interderivable with the sentence $\ulcorner u \urcorner$ *is not a theorem of* $\mathbb{ZF}$. Then for each $y \in S$, the formula $g(\ulcorner y \urcorner)$ *is not a proof of* $\ulcorner u \urcorner$ is a theorem of $\mathbb{ZF}$. This implies that $g(\mathfrak{y})$ *is not a proof of* $\ulcorner u \urcorner$ follows from $\mathfrak{y} = \ulcorner y \urcorner$ for each $y \in S$; and hence from $\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\}$. But if $g(\mathfrak{y})$ *is not a proof of* $\ulcorner u \urcorner$ were a theorem, so would be its universal generalization $\ulcorner u \urcorner$ *is not a theorem of* $\mathbb{ZF}$, and so also would be the sentence u.

Theorem 28 can be considerably improved if we choose for **Cn** a logic based on an ω-consistent and incompletable theory such as elementary Peano arithmetic. In such logics the set **L** of theorems is (we hope) ω-consistent, but incompletable in the sense that there exists no finitely (or even recursively) axiomatizable theory that is maximal. We begin with a simplifying lemma.

LEMMA 29 Suppose that **Cn** is ω-consistent. If $\boldsymbol{\delta}(\mathfrak{y}) \vdash w(\mathfrak{y})$, then $\forall \mathfrak{y} w(\mathfrak{y})$ is consistent.

PROOF By (2.1), $w(\mathfrak{y})$ is implied by some finite conjunction $z_0(\mathfrak{y}) \wedge \cdots \wedge z_{k-1}(\mathfrak{y})$ of $\mathfrak{y}$-universal formulas, and hence $\forall\mathfrak{y}z_0(\mathfrak{y}) \wedge \cdots \wedge \forall\mathfrak{y}z_{k-1}(\mathfrak{y}) \vdash \forall\mathfrak{y}w(\mathfrak{y})$. In an ω-consistent theory, the universal quantification of a $\mathfrak{y}$-universal formula is consistent; ergo, $\forall\mathfrak{y}w(\mathfrak{y})$ is consistent (in **Cn**). ∎

THEOREM 30 If **Cn** is ω-consistent and incompletable then the theory $\boldsymbol{\delta}(\mathfrak{y})$ is unaxiomatizable.

PROOF Suppose that there is some formula $w(\mathfrak{y})$ for which $\boldsymbol{\delta}(\mathfrak{y}) = \mathbf{Cn}(w(\mathfrak{y}))$. By Lemma 29, the sentence $u = \forall\mathfrak{y}w(\mathfrak{y})$ is consistent in **Cn**. It will be shown that $\mathbf{Cn}(u)$ is a maximal theory.

The quantifier-free sentences of the language of Peano arithmetic are all decidable (either such a sentence x is a theorem, or its negation $\neg x$ is a theorem). If the extended language $\mathcal{L}^+$ contains quantifier-free sentences that are not decidable, then one of each pair $\{x, \neg x\}$ may be added in a constructive manner to the stock of theorems. In other words, we may safely assume that if $x(\mathfrak{y})$ is a formula of one free variable, then u decides each quantifier-free sentence $x(\ulcorner y \urcorner)$.

Now let $x(\mathfrak{y})$ be a formula that contains free at least the variable $\mathfrak{y}$. If $u \vdash x(\ulcorner y \urcorner)$ for some y, then $u \vdash \exists\mathfrak{y}x(\mathfrak{y})$. The alternative is that $u \vdash \neg x(\ulcorner y \urcorner)$ for every y, and so $u \rightarrow \neg x(\ulcorner y \urcorner) \in \mathbf{L}$ for every y, and hence $u \rightarrow \neg x(\mathfrak{y})$ is $\mathfrak{y}$-universal. It follows that $w(\mathfrak{y}) \vdash u \rightarrow \neg x(\mathfrak{y})$. But $u \vdash w(\mathfrak{y})$, and hence $u \vdash \neg x(\mathfrak{y})$. Since u is a sentence, $u \vdash \forall\mathfrak{y}\neg x(\mathfrak{y})$, so $u \vdash \neg\exists\mathfrak{y}x(\mathfrak{y})$.

For all formulas x containing $\mathfrak{y}$ free, that is to say, u decides $\exists\mathfrak{y}x(\mathfrak{y})$ if it decides each instance $x(\ulcorner y \urcorner)$, and so u decides also $\exists\mathfrak{y}\neg x(\mathfrak{y})$, and hence also $\forall\mathfrak{y}x(\mathfrak{y})$. This means that a universal formula $\forall\mathfrak{y}x(\mathfrak{y})$ is decided by u if all its instances $x(\ulcorner y \urcorner)$ are. The base of the induction was settled two paragraphs ago, and we may therefore conclude that every universal formula is decided by u. In short, $\mathbf{Cn}(u)$ is a maximal theory. This is impossible if **Cn** is incompletable. ∎

Theorems 24, 25, and 30 establish the unaxiomatizability, under appropriate conditions, of the theories $\boldsymbol{\vartheta}(\mathfrak{y})$, $\boldsymbol{\theta}(\mathfrak{y})$, $\boldsymbol{\delta}(\mathfrak{y})$, where $\mathfrak{y}$ is a variable for names of formulas. With care they can be extended to other terms, including most names for formulas that are not structural-descriptive names. By Lemma 23, Theorem 24, and Theorem 25, for example, the theories $\boldsymbol{\theta}(\mathfrak{p})$ and $\boldsymbol{\vartheta}(\mathfrak{p})$, where $\mathfrak{p}$ = 'the first sentence of *Pride & Prejudice*', are unaxiomatizable in classical logic, provided that there are infinitely many non-theorems in the set $\{\mathfrak{p} = \ulcorner y \urcorner \rightarrow y) \mid y \in S\}$.

6 Falsehood and Untruth

In this section we assume that the consequence operation **Cn** is non-trivial and includes the whole of classical logic. Our task is to establish that, undeterred by their classical origins, the function $\boldsymbol{\vartheta}$ exhibits truth-value gaps and the functions $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$ both exhibit truth-value gluts. In parallel with the use of the two lower-case forms of the Greek letter *theta*, $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$, for the two functions defining truth, we shall henceforth use the two lower-case forms of *phi*, $\boldsymbol{\varphi}$ and $\boldsymbol{\phi}$, for the corresponding functions defining falsity, and the two lower-case forms of *pi*, $\boldsymbol{\varpi}$ and $\boldsymbol{\pi}$, for the corresponding functions defining untruth.

The falsity of a formula, anyway in classical logic, should be little more than the truth of its negation. One way to express this identity is to employ, (or, if necessary, introduce), for any named formula, a name for the formula that would be obtained by some canonical insertion of a negation sign. But rather than take such a route, which will be taken in Miller (2009), when we deal systematically with questions about compositionality, we shall proceed as follows. We define

$$\boldsymbol{\varphi}(\mathfrak{y}) \quad = \quad \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \wedge \neg y) \mid y \in S\}, \tag{6.0}$$

$$\boldsymbol{\phi}(\mathfrak{y}) \quad = \quad \bigwedge\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to \neg y) \mid y \in S\}, \tag{6.1}$$

as the *falsity* operations that correspond to $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$, and alongside them define also two *untruth* (or *perjury*) operations $\boldsymbol{\varpi}(\mathfrak{y})$ and $\boldsymbol{\pi}(\mathfrak{y})$ by means of theory complementation (4.3):

$$\boldsymbol{\varpi}(\mathfrak{y}) \quad =_{\mathrm{Df}} \quad \boldsymbol{\vartheta}(\mathfrak{y})', \tag{6.2}$$

$$\boldsymbol{\pi}(\mathfrak{y}) \quad =_{\mathrm{Df}} \quad \boldsymbol{\theta}(\mathfrak{y})'. \tag{6.3}$$

Theorem 31 For classical **Cn**, (6.0) and (6.2), and (6.1) and (6.3), are materially adequate:

$$\boldsymbol{\varphi}(\ulcorner x \urcorner) = \quad \mathbf{Cn}(\neg x) \quad = \boldsymbol{\varpi}(\ulcorner x \urcorner), \tag{6.4}$$

$$\boldsymbol{\phi}(\ulcorner x \urcorner) = \quad \mathbf{Cn}(\neg x) \quad = \boldsymbol{\pi}(\ulcorner x \urcorner), \tag{6.5}$$

whenever $\ulcorner x \urcorner$ is the structural-descriptive name of a formula $x \in S$.

Proof The proof is immediate, using the assumed properties of structural-descriptive names. ■

Lemma 32 If **Cn** includes classical logic, and $\mathbf{Y}'$ stands for the complement of the theory $\mathbf{Y}$,

$$\boldsymbol{\varpi}(\mathfrak{y}) \quad = \quad \boldsymbol{\varphi}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y})', \tag{6.6}$$

$$\boldsymbol{\pi}(\mathfrak{y}) \quad = \quad \boldsymbol{\varphi}(\mathfrak{y}) \quad = \quad \boldsymbol{\phi}(\mathfrak{y}) \wedge \boldsymbol{\delta}(\mathfrak{y}). \tag{6.7}$$

Proof By (3.4), $\boldsymbol{\vartheta}(\mathfrak{y}) = \boldsymbol{\theta}(\mathfrak{y}) \wedge \boldsymbol{\delta}(\mathfrak{y})$, and so by (3.0), (4.9), Lemma 17, (6.2), and (6.0),

$$\begin{aligned}
\boldsymbol{\varpi}(\mathfrak{y}) \quad = \quad \boldsymbol{\vartheta}(\mathfrak{y})' \quad &= \quad (\bigwedge\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to y) \mid y \in S\} \wedge \boldsymbol{\delta}(\mathfrak{y}))' \\
&= \quad \bigvee\{(\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to y))' \mid y \in S\} \vee \boldsymbol{\delta}(\mathfrak{y})' \\
&= \quad \bigvee\{\mathbf{Cn}(\neg(\mathfrak{y} = \ulcorner y \urcorner \to y)) \mid y \in S\} \vee \boldsymbol{\delta}(\mathfrak{y})' \\
&= \quad \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \wedge \neg y) \mid y \in S\} \vee \boldsymbol{\delta}(\mathfrak{y})' \\
&= \quad \boldsymbol{\varphi}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y})'.
\end{aligned}$$

The identity of $\boldsymbol{\pi}(\mathfrak{y})$ and $\boldsymbol{\varphi}(\mathfrak{y})$ is a simple consequence of (6.2), (3.1), (4.9), and (6.0). To prove that $\boldsymbol{\varphi}(\mathfrak{y}) = \boldsymbol{\phi}(\mathfrak{y}) \wedge \boldsymbol{\delta}(\mathfrak{y})$, it should suffice to repeat the proof of Theorem 3.4. ■

According to (6.6) and (6.7), $\boldsymbol{\varphi}(\mathfrak{y})$ implies $\boldsymbol{\varpi}(\mathfrak{y})$, and $\boldsymbol{\pi}(\mathfrak{y})$ implies $\boldsymbol{\phi}(\mathfrak{y})$. For structural-descriptive names, falsity and untruth coincide, but in general the implications are not reversible.

Theorem 33 If the logic **Cn** is ω-incomplete (that is, according to Theorem 28, $\boldsymbol{\delta}(\mathfrak{y}) \neq \mathbf{L}$), then $\boldsymbol{\varpi}(\mathfrak{y})$ does not imply $\boldsymbol{\varphi}(\mathfrak{y})$, and $\boldsymbol{\phi}(\mathfrak{y})$ does not imply $\boldsymbol{\pi}(\mathfrak{y})$.

Proof If $\boldsymbol{\varpi}(\mathfrak{y})$ implies $\boldsymbol{\varphi}(\mathfrak{y})$, then by (6.6), $\boldsymbol{\delta}(\mathfrak{y})'$ implies $\boldsymbol{\varphi}(\mathfrak{y})$. It is evident from their definitions (3.2) and (6.0) that $\boldsymbol{\delta}(\mathfrak{y})$, which is identical with $\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner) \mid y \in S\}$, is implied by $\boldsymbol{\varphi}(\mathfrak{y})$, which is identical with $\bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \wedge \neg y) \mid y \in S\}$, and hence that $\boldsymbol{\delta}(\mathfrak{y}) \vee \boldsymbol{\varphi}(\mathfrak{y}) = \boldsymbol{\delta}(\mathfrak{y})$. It follows that $\boldsymbol{\delta}(\mathfrak{y}) \vee \boldsymbol{\delta}(\mathfrak{y})'$ implies $\boldsymbol{\delta}(\mathfrak{y})$, and so, by (4.4), $\boldsymbol{\delta}(\mathfrak{y}) = \mathbf{L}$.

If $\boldsymbol{\phi}(\mathfrak{y})$ implies $\boldsymbol{\pi}(\mathfrak{y})$, then by (6.7) and Lemma 26, $\boldsymbol{\phi}(\mathfrak{y})$ implies every $\mathfrak{y}$-universal formula $z(\mathfrak{y})$. By (2.1), there is, for each $z(\mathfrak{y})$, some finite set $\{y_i \mid i < k\}$ of formulas such that the conjunction $(\mathfrak{y} = \ulcorner y_0 \urcorner \to \neg y_0) \wedge \cdots \wedge (\mathfrak{y} = \ulcorner y_{k-1} \urcorner \to \neg y_{k-1})$ implies $z(\mathfrak{y})$. It follows that the conjunction $\mathfrak{y} \neq \ulcorner y_0 \urcorner \wedge \cdots \wedge \mathfrak{y} \neq \ulcorner y_{k-1} \urcorner$ implies $z(\mathfrak{y})$. But since $z(\mathfrak{y})$ is $\mathfrak{y}$-universal, the finite disjunction $\mathfrak{y} = \ulcorner y_0 \urcorner \vee \cdots \vee \mathfrak{y} = \ulcorner y_{k-1} \urcorner$ also implies $z(\mathfrak{y})$. In short, every $\mathfrak{y}$-universal formula $z(\mathfrak{y})$ is demonstrable. This cannot be the case if $\boldsymbol{\delta}(\mathfrak{y}) \neq \mathbf{L}$, as has been assumed. ■

Lemma 34 If the logic **Cn** is ω-incomplete (more generally, if $\boldsymbol{\delta}(\mathfrak{y}) \neq \mathbf{L}$), then $\boldsymbol{\vartheta}(\mathfrak{y}) \vee \boldsymbol{\varphi}(\mathfrak{y}) \neq \mathbf{L}$.

Proof If both $\boldsymbol{\vartheta}(\mathfrak{y})$ and $\boldsymbol{\varphi}(\mathfrak{y})$ imply the formula x, then each formula of the form $\mathfrak{y} = \ulcorner y \urcorner \wedge y$ implies x and each formula of the form $\mathfrak{y} = \ulcorner y \urcorner \wedge \neg y$ implies x. It follows that each formula $\mathfrak{y} = \ulcorner y \urcorner$ implies x, and hence that $\boldsymbol{\delta}(\mathfrak{y})$ implies x. If $\boldsymbol{\delta}(\mathfrak{y}) \neq \mathbf{L}$ then $\boldsymbol{\vartheta}(\mathfrak{y}) \vee \boldsymbol{\varphi}(\mathfrak{y}) \neq \mathbf{L}$. ■

Lemma 35 If the logic **Cn** is ω-consistent and incompletable (more generally, if $\boldsymbol{\delta}(\mathfrak{y})$ is unaxiomatizable), then $\boldsymbol{\vartheta}(\mathfrak{y}) \wedge \boldsymbol{\varphi}(\mathfrak{y}) \neq \mathbf{S}$.

Proof By (3.0), (6.0), and Theorem 3 we have for any $x \in S$

$$\begin{aligned} \boldsymbol{\vartheta}(\mathfrak{y}) \wedge \boldsymbol{\varphi}(\mathfrak{y}) &= \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \wedge y) \mid y \in S\} \wedge \\ &\quad \wedge \bigvee\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \wedge \neg y) \mid y \in S\} \\ &= \bigwedge\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to y) \mid y \in S\} \wedge \boldsymbol{\delta}(\mathfrak{y}) \wedge \\ &\quad \wedge \bigwedge\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y \urcorner \to \neg y) \mid y \in S\} \wedge \boldsymbol{\delta}(\mathfrak{y}). \end{aligned} \tag{6.8}$$

Now $(\mathfrak{y} = \ulcorner y \urcorner \to y) \wedge (\mathfrak{y} = \ulcorner y \urcorner \to \neg y)$ is logically equivalent to $\mathfrak{y} \neq \ulcorner y \urcorner$. It follows that $\boldsymbol{\vartheta}(\mathfrak{y}) \wedge \boldsymbol{\varphi}(\mathfrak{y}) = \bigwedge\{\mathfrak{y} \neq \ulcorner y \urcorner \mid y \in S\} \wedge \boldsymbol{\delta}(\mathfrak{y}) = \bigwedge\{\mathfrak{y} \neq \ulcorner y \urcorner \mid y \in S\} \wedge \bigvee\{\mathfrak{y} = \ulcorner y \urcorner \mid y \in S\}$, which by (4.9), is identical with $\bigwedge\{\mathfrak{y} \neq \ulcorner y \urcorner \mid y \in S\} \wedge (\bigwedge\{\mathfrak{y} \neq \ulcorner y \urcorner \mid y \in S\})'$. By Theorem 18, $\boldsymbol{\vartheta}(\mathfrak{y}) \wedge \boldsymbol{\varphi}(\mathfrak{y}) = \mathbf{S}$ if & only if $\bigwedge\{\mathfrak{y} \neq \ulcorner y \urcorner \mid y \in S\}$ is axiomatizable. In classical logic, the complement $\mathbf{Y}'$ of an axiomatizable theory $\mathbf{Y}$ is also axiomatizable. It follows that if $\boldsymbol{\vartheta}(\mathfrak{y}) \wedge \boldsymbol{\varphi}(\mathfrak{y}) = \mathbf{S}$ then $\boldsymbol{\delta}(\mathfrak{y}) = (\bigwedge\{\mathfrak{y} \neq \ulcorner y \urcorner \mid y \in S\})'$ is axiomatizable. The result follows by contraposition. ■

Theorem 36 In general, $\boldsymbol{\vartheta}(\mathfrak{y}) \vee \boldsymbol{\varpi}(\mathfrak{y}) = \boldsymbol{\theta}(\mathfrak{y}) \vee \boldsymbol{\pi}(\mathfrak{y}) = \boldsymbol{\theta}(\mathfrak{y}) \vee \boldsymbol{\phi}(\mathfrak{y}) = \mathbf{L}$, but $\boldsymbol{\vartheta}(\mathfrak{y}) \vee \boldsymbol{\varphi}(\mathfrak{y}) \neq \mathbf{L}$.

Proof The law of excluded middle (4.4) holds for all deductive theories, and hence $\boldsymbol{\vartheta}(\mathfrak{y}) \vee \boldsymbol{\varpi}(\mathfrak{y})$ and $\boldsymbol{\theta}(\mathfrak{y}) \vee \boldsymbol{\pi}(\mathfrak{y})$ are both $\mathbf{L}$. By (6.7), $\boldsymbol{\theta}(\mathfrak{y}) \vee \boldsymbol{\phi}(\mathfrak{y}) = \mathbf{L}$. By Lemma 34, $\boldsymbol{\vartheta}(\mathfrak{y}) \vee \boldsymbol{\varphi}(\mathfrak{y}) \neq \mathbf{L}$. ■

THEOREM 37 In general, none of the four theories $\boldsymbol{\vartheta}(\mathfrak{y}) \wedge \boldsymbol{\varpi}(\mathfrak{y})$, $\boldsymbol{\theta}(\mathfrak{y}) \wedge \boldsymbol{\pi}(\mathfrak{y})$, $\boldsymbol{\theta}(\mathfrak{y}) \wedge \boldsymbol{\phi}(\mathfrak{y})$, $\boldsymbol{\vartheta}(\mathfrak{y}) \wedge \boldsymbol{\varphi}(\mathfrak{y})$ is identical with S.

PROOF By Theorems 18, 24, and 25, $\boldsymbol{\vartheta}(\mathfrak{y}) \wedge \boldsymbol{\varpi}(\mathfrak{y}) \neq \mathsf{S}$ and $\boldsymbol{\theta}(\mathfrak{y}) \wedge \boldsymbol{\pi}(\mathfrak{y}) \neq \mathsf{S}$. It then follows from (6.7) that $\boldsymbol{\theta}(\mathfrak{y}) \wedge \boldsymbol{\phi}(\mathfrak{y}) \neq \mathsf{S}$, and Lemma 35 says that $\boldsymbol{\vartheta}(\mathfrak{y}) \wedge \boldsymbol{\varphi}(\mathfrak{y})$ may differ from S. ■

Loosely stated: only $\boldsymbol{\vartheta}$ admits *truth-value gaps*, but both $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$ admit *truth-value gluts*.

7 The Truth of Theories

A deductive theory $\mathsf{Y} = \mathsf{Cn}(\mathsf{Y})$ is true if & only if every sentence y in Y is true. That is what we are tempted to say. But on the present account, truth is not a predicate, and we cannot say what we are tempted to say. Yet the truth of a sentence is a theory, and the truth of that theory is presumably no more than the truth of the sentence. If $\mathfrak{Y}$ is an expression (variable or name) that stands for a theory, or more generally any set of formulas of $\mathcal{L}^+$, our aim is to extend the definitions of $\boldsymbol{\vartheta}(\mathfrak{y})$ and $\boldsymbol{\theta}(\mathfrak{y})$ in such a way that if $\mathfrak{Y} = \mathsf{Cn}(\mathfrak{y})$ is demonstrable in the background theory $\mathbb{T}$ then $\boldsymbol{\vartheta}(\mathfrak{Y}) = \boldsymbol{\vartheta}(\mathfrak{y})$ and $\boldsymbol{\theta}(\mathfrak{Y}) = \boldsymbol{\theta}(\mathfrak{y})$ are also demonstrable. Like $\boldsymbol{\vartheta}(\mathfrak{y})$ and $\boldsymbol{\theta}(\mathfrak{y})$, the sets $\boldsymbol{\vartheta}(\mathfrak{Y})$ and $\boldsymbol{\theta}(\mathfrak{Y})$ are to be deductive theories.

The least that we must hope for in addition is that the extended definition of truth be materially adequate. Theorem 38 of Tarski (1935–1936) states that in a non-trivial classical deductive system there are continuum many deductive theories, which implies that only a handful of them, in addition to the axiomatizable theories, can be endowed with structural-descriptive names. It is perhaps reasonable to hope that a structural-descriptive name $\ulcorner\mathsf{Y}\urcorner$ can be given to each recursively axiomatizable theory Y. But if we insist that the primary demands on structural-descriptive names, that $\ulcorner\mathsf{X}\urcorner = \ulcorner\mathsf{Z}\urcorner$ be demonstrable if $\mathsf{X} = \mathsf{Z}$ and refutable if $\mathsf{X} \neq \mathsf{Z}$, and that $\ulcorner x\urcorner \in \ulcorner\mathsf{Z}\urcorner$ be demonstrable if $x \in \mathsf{Z}$ and refutable if $x \notin \mathsf{Z}$, be achieved within the logic Cn, then the investigation will have to be restricted to logics sufficiently powerful to allow this.

The line of approach to be adopted here has recourse to a simple extension of Tarski's calculus of deductive theories. It is disappointing that, although it works well for $\boldsymbol{\theta}$, it fails badly for $\boldsymbol{\vartheta}$.

We have seen in (2.3)–(2.6) how Tarski proposed to define finite and infinite analogues of Boolean operations on theories. To my knowledge he never explicitly considered defining analogues of quantification, though he explored in detail elsewhere the closely related operation of cylindrification (for a brief survey of his work on cylindric algebras see Monk 1986, pp. 902–905).

Let Y be a theory (or any other set of formulas drawn from S) in which $\mathfrak{y}$ is a variable that is sometimes free. We shall define $\exists\mathfrak{y}\mathsf{Y}$ and $\forall\mathfrak{y}\mathsf{Y}$ as theories, and then extend the definition of truth $\boldsymbol{\theta}$ from formulas to theories so as to deliver the wanted analogue of material adequacy. But it does not appear to be possible to obtain a corresponding result for the function $\boldsymbol{\vartheta}$.

Every theory Y contains formulas (for example, $\mathfrak{y} = \mathfrak{y}$) in which the variable $\mathfrak{y}$ occurs free. We are interested at present in those theories in which $\mathfrak{y}$ has no essential occurrence. Y will accordingly be called $\mathfrak{y}$-*less* (rather than $\mathfrak{y}$-free, which might be too confusing) if $\mathbf{Y} = \mathbf{Cn}(Y)$ for some set Y none of whose elements contains $\mathfrak{y}$ free. We proceed to define, rather obviously,

$$\exists\mathfrak{y}\mathbf{Y} \quad =_{\mathrm{Df}} \quad \bigwedge\{\mathbf{X} \mid \mathbf{X} \text{ is } \mathfrak{y}\text{-less and } \mathbf{Y} \vdash \mathbf{X}\}, \tag{7.0}$$

$$\forall\mathfrak{y}\mathbf{Y} \quad =_{\mathrm{Df}} \quad \bigvee\{\mathbf{X} \mid \mathbf{X} \text{ is } \mathfrak{y}\text{-less and } \mathbf{X} \vdash \mathbf{Y}\}. \tag{7.1}$$

Taking for granted that these definitions license the usual quantifier inferences, we may define:

$$\boldsymbol{\vartheta}(\mathfrak{Y}) \quad =_{\mathrm{Df}} \quad \forall\mathfrak{y}[\mathbf{Cn}(\mathfrak{y} \in \mathfrak{Y}) \twoheadrightarrow \boldsymbol{\vartheta}(\mathfrak{y})], \tag{7.2}$$

$$\boldsymbol{\theta}(\mathfrak{Y}) \quad =_{\mathrm{Df}} \quad \forall\mathfrak{y}[\mathbf{Cn}(\mathfrak{y} \in \mathfrak{Y}) \twoheadrightarrow \boldsymbol{\theta}(\mathfrak{y})]. \tag{7.3}$$

It will be recalled from Lemma 19 that in classical logic the conditional $\mathbf{X} \twoheadrightarrow \mathbf{Z}$ exists whenever $\mathbf{X}$ is axiomatizable, and is identical with $\mathbf{X}' \vee \mathbf{Z}$; if $\mathbf{X} = \mathbf{Cn}(x)$ then $\mathbf{X} \twoheadrightarrow \mathbf{Z} = \neg x \vee \mathbf{Z}$. Similar extensions to deductive theories may be given for the functions $\boldsymbol{\varphi}$, $\boldsymbol{\phi}$, $\boldsymbol{\varpi}$, and $\boldsymbol{\pi}$.

There exist direct generalizations of Theorem 2 for $\boldsymbol{\theta}$ and $\boldsymbol{\vartheta}$, and of Theorem 3.3 for $\boldsymbol{\theta}$.

THEOREM 38 If the theory named by $\mathfrak{X}$ is implied in $\mathbf{Cn}$ by the theory named by $\mathfrak{Z}$, then $\boldsymbol{\theta}(\mathfrak{X})$ is implied by $\boldsymbol{\theta}(\mathfrak{Z})$.

PROOF Immediate using (2.6). ■

THEOREM 39 If $\ulcorner\mathbf{Y}\urcorner$ is a structural-descriptive name for the $\mathfrak{y}$-less theory $\mathbf{Y}$, then

$$\boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner) \quad = \quad \mathbf{Y}. \tag{7.4}$$

PROOF If $y \in \mathbf{Y}$, then the formula $\ulcorner y\urcorner \in \ulcorner\mathbf{Y}\urcorner$ is demonstrable in $\mathbf{Cn}$, and hence the theory $\mathbf{Cn}(\ulcorner y\urcorner \in \ulcorner\mathbf{Y}\urcorner) \twoheadrightarrow \boldsymbol{\theta}(\ulcorner y\urcorner)$, which is identical with the theory $\mathbf{Cn}(\ulcorner y\urcorner \notin \ulcorner\mathbf{Y}\urcorner) \vee \boldsymbol{\theta}(\ulcorner y\urcorner)$, is demonstrably identical with $\boldsymbol{\theta}(\ulcorner y\urcorner)$. Since, by (7.3), $\boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner)$ implies the theory $\mathbf{Cn}(\ulcorner y\urcorner \in \ulcorner\mathbf{Y}\urcorner) \twoheadrightarrow \boldsymbol{\theta}(\ulcorner y\urcorner)$ for every $y \in S$, it implies $\boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner)$ for every $y \in \mathbf{Y}$. By Theorem 1, $\boldsymbol{\theta}(\ulcorner y\urcorner) = \mathbf{Cn}(y)$. In short, $\boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner)$ implies $\mathbf{Y}$. (If $y \notin \mathbf{Y}$ then $\ulcorner y\urcorner \in \ulcorner\mathbf{Y}\urcorner$ is refutable, and $\mathbf{Cn}(\ulcorner y\urcorner \in \ulcorner\mathbf{Y}\urcorner) \twoheadrightarrow \boldsymbol{\theta}(\ulcorner y\urcorner) = \mathbf{L}$.)

For the converse implication, note that if $y \in \mathbf{Y}$ then $\mathbf{Y}$ implies y, while if $y \notin \mathbf{Y}$ then $\mathbf{Cn}(\ulcorner y\urcorner \in \ulcorner\mathbf{Y}\urcorner) = \mathbf{S}$. In other words, for each $y \in S$, the theory $\mathbf{Y} \wedge \mathbf{Cn}(\ulcorner y\urcorner \in \ulcorner\mathbf{Y}\urcorner)$ implies y, and therefore, for each $y \in S$, the theory $\mathbf{Y} \wedge \mathbf{Cn}(\mathfrak{y} \in \ulcorner\mathbf{Y}\urcorner) \wedge \mathbf{Cn}(\mathfrak{y} = \ulcorner y\urcorner)$ implies y. By Lemma 19, $\mathbf{Y} \wedge \mathbf{Cn}(\mathfrak{y} \in \ulcorner\mathbf{Y}\urcorner)$ implies every (axiomatizable) theory $\mathbf{Cn}(\mathfrak{y} = \ulcorner y\urcorner) \twoheadrightarrow y$, where $y \in S$, and therefore implies their meet $\bigwedge\{\mathbf{Cn}(\mathfrak{y} = \ulcorner y\urcorner \rightarrow y) \mid y \in S\}$, which is $\boldsymbol{\theta}(\mathfrak{y})$. By Lemma 19 again, $\mathbf{Y}$ implies the theory $\mathbf{Cn}(\mathfrak{y} \in \ulcorner\mathbf{Y}\urcorner) \twoheadrightarrow \boldsymbol{\theta}(\mathfrak{y})$. Since $\mathbf{Y}$ is $\mathfrak{y}$-less, we may conclude that $\mathbf{Y}$ implies $\boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner)$. ■

THEOREM 40 If $\ulcorner\mathbf{Y}\urcorner$ is a structural-descriptive name for the $\mathfrak{y}$-less theory $\mathbf{Y}$, then

$$\boldsymbol{\vartheta}(\ulcorner\mathbf{Y}\urcorner) \quad \vdash \quad \mathbf{Y}. \tag{7.5}$$

PROOF By the definition (7.2), $\boldsymbol{\vartheta}(\ulcorner\mathbf{Y}\urcorner)$ implies $\mathbf{Cn}(\ulcorner y\urcorner \in \ulcorner\mathbf{Y}\urcorner) \twoheadrightarrow \boldsymbol{\vartheta}(\ulcorner y\urcorner)$ for each y; and, on the assumption (made use of throughout the previous theorem) that formulas such as $\ulcorner y\urcorner \in \ulcorner\mathbf{Y}\urcorner$ are correctly decidable in $\mathbf{Cn}$, it follows that

$\boldsymbol{\vartheta}(\ulcorner\mathbf{Y}\urcorner)$ implies $\boldsymbol{\vartheta}(\ulcorner y\urcorner)$ for each $y \in \mathbf{Y}$. By (3.3), $\boldsymbol{\vartheta}(\ulcorner\mathbf{Y}\urcorner)$ implies $\mathbf{Cn}(y)$ for each $y \in \mathbf{Y}$; that is, $\boldsymbol{\vartheta}(\ulcorner\mathbf{Y}\urcorner) \vdash \mathbf{Y}$. ■

Let us inquire a little further why the converse of Theorem 40 is unattainable. If $\forall\mathfrak{y}[\mathbf{Cn}(\mathfrak{y} \in \ulcorner\mathbf{Y}\urcorner) \twoheadrightarrow \boldsymbol{\vartheta}(\mathfrak{y})]$ were implied by $\mathbf{Y}$, then $\boldsymbol{\vartheta}(\mathfrak{y})$ would be implied by $\mathbf{Y} \wedge \mathbf{Cn}(\mathfrak{y} \in \ulcorner\mathbf{Y}\urcorner)$. But then by (3.4), $\boldsymbol{\theta}(\mathfrak{y}) \wedge \boldsymbol{\delta}(\mathfrak{y})$ would be so implied. The first conjunct is unproblematic. But write $\mathfrak{p}$ for $\mathfrak{y}$, and let $\mathbf{Y}$ be $\mathbf{Cn}(p)$. Then $\mathbf{Cn}(\mathfrak{p} \in \ulcorner\mathbf{Y}\urcorner) = \mathbf{L}$, and hence $\mathbf{Cn}(p)$ implies $\boldsymbol{\delta}(\mathfrak{p})$; which means, by (26), that the sentence 'It is a truth universally acknowledged, that a single man in possession of a good fortune, must be in want of a wife', which we called 'p', implies $z(\mathfrak{p})$ where z is any $\mathfrak{y}$-universal formula of S. It is one thing for $\boldsymbol{\vartheta}(\mathfrak{p})$ to imply $z(\mathfrak{p})$, but quite another thing for p to do so. If the implication were to hold, indeed, the definition (7.2) would be a creative one.

These results (Theorem 39, and the absence of a corresponding identity for $\boldsymbol{\vartheta}$) seem to settle decisively the question of which theory, $\boldsymbol{\vartheta}(\mathfrak{y})$ or $\boldsymbol{\theta}(\mathfrak{y})$, is to be preferred for the task of defining the truth of the formula named by $\mathfrak{y}$. As was made clear in § 3, the function $\boldsymbol{\vartheta}$ is much more generally applicable amongst weak deductive disciplines. But in disciplines such as arithmetic and set theory, where unaxiomatizable theories abound, the function $\boldsymbol{\theta}$, unlike $\boldsymbol{\vartheta}$, provides a definition of truth that meets the minimum standard of material adequacy. To avoid minor irregularities, noted just before (3.0) above, the domain of the function $\boldsymbol{\theta}$ must be restricted to names of formulas (and theories). Within that domain it succeeds in doing a satisfactory job.

8 The Antinomy of the Liar

Having introduced no truth predicate, the present discussion is immune from the liar antinomy in its customary forms. Although we have presented two materially adequate definitions of truth, $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$, that assign to each name of a formula a theory that encapsulates its truth, it is plain that there can exist at best an axiomatizable theory, but never a sentence or formula, that asserts its own untruth. The liar antinomy beloved of philosophers is unceremoniously blocked.

The antinomy seems to be reborn, however, if a theory $\mathbf{Y}$ can be constructed for which $\mathbf{Y} = \boldsymbol{\pi}(\ulcorner\mathbf{Y}\urcorner) = \boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner)'$. For according to Theorem 39, $\mathbf{Y} = \boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner)$, and according to (4.4), $\boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner) \vee \boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner)' = \mathbf{L}$. It follows that both $\boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner)$ and its complement $\boldsymbol{\theta}(\ulcorner\mathbf{Y}\urcorner)'$ are identical with $\mathbf{L}$, and therefore that $\mathbf{L} = \mathbf{S}$. The antinomy of the liar is not expunged merely by the adoption of a paraconsistent logic in which truth and falsity (and truth and untruth) are not always incompatible (Theorem 37). The version of the liar just sketched presumes to exhibit a theory that is not just both true and false but both logically true and logically false.

This cannot, of course, be right, since the definition (7.3) of $\boldsymbol{\theta}(\mathfrak{Y})$, and its ancestors, are all explicit definitions within the background theory $\mathbb{T}$ (which is surely consistent). It is simple enough to pinpoint the invalid step: it must lie in the supposed construction of the theory $\mathbf{Y} = \boldsymbol{\pi}(\ulcorner\mathbf{Y}\urcorner)$. For although it is perfectly possible, by parodying its definition, to provide a theory such as $\boldsymbol{\pi}(\mathfrak{Y})$ with a name, $^{\langle}\boldsymbol{\pi}(\mathfrak{Y})^{\rangle}$ say, that in some sense reveals its structure, and then apply the diagonal construction, such names are not of a kind that permits proof of an analogue of

(7.4), that $\mathbf{Y} = \boldsymbol{\theta}(\langle Y \rangle)$. To take a simple example, consider $\boldsymbol{\pi}(\mathfrak{y})$, which is defined in (6.3) as $\boldsymbol{\theta}(\mathfrak{y})'$, or equivalently (by (4.9) and (3.1)) as $\bigvee\{\mathfrak{y} = \ulcorner y \urcorner \wedge \neg y \mid y \in S\}$. The structure of this theory — let us call it $\mathbf{Z}$ — is not well enough known for a name $\langle \mathbf{Z} \rangle$ to provide information about which formulas $\mathbf{Z}$ does and does not imply. We cannot expect, that is, to prove $\ulcorner y \urcorner \in \langle \mathbf{Z} \rangle$ when $\mathbf{Z}$ implies y, or (especially) to prove $\ulcorner y \urcorner \notin \langle \mathbf{Z} \rangle$ when $\mathbf{Z}$ does not imply y. But it is exactly these features of names of formulas that are needed for material adequacy (7.4) to be established.

The above three paragraphs give only a hint of why and how the antinomy of the liar is bypassed in this theory of truth. A more elaborate treatment is planned for Miller (2009).

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