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Dynamic copula quantile regressions and tail area dynamic dependence in Forex markets

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We introduce a general approach to nonlinear quantile regression modelling based on the copula function that defines the dependency structure between the variables of interest. Hence, we extend Koenker and Bassett's (1978. Regression quantiles. *Econometrica*, 46, no. 1: 33–50.) original statement of the quantile regression problem by determining a distribution for the dependent variable Y conditional on the regressors X, and hence the specification of the quantile regression functions. The approach exploits the fact that the joint distribution function can be split into two parts: the marginals and the dependence function (or copula). We then deduce the form of the (invariably nonlinear) conditional quantile relationship implied by the copula. This can be achieved with arbitrary distributions assumed for the marginals. Some properties of the copula-based quantiles or c-quantiles are derived. Finally, we examine the conditional quantile dependency in the foreign exchange market and compare our quantile approach with standard tail area dependency measures.

Keywords: Copula; Quantile; Regression; dependence; foreign exchange markets

1. Introduction

The problem of characterizing the conditional dependence between random variables at a given quantile is an important practical issue in risk management and portfolio design. It is also one which is difficult to address if the joint distribution of the variables involved is nonelliptic and fat tailed as is standard with financial returns. Tail area dependency for instance may be quite different to that implied by correlation and may signal where downside protection can be found if two assets do not show positive causal dependency in their extreme quantiles. One goal of this paper is to introduce a general approach to this problem through nonlinear quantile regression modelling where the form of the quantile regression is implied by the copula linking the assets involved.

Our starting point is the joint distribution for the variables which will almost certainly be non-Gaussian. Working down, in a general to specific manner, this multivariate distribution can be split into two parts – the marginal densities and the dependence function or copula that joins these marginals together to give the joint distribution function. Since the copula function holds all information on the different forms of dependency that exist between the assets, we can see that the form of the conditional quantile relationship is implied by the copula function. We refer below to this relationship as a *copula-quantile regression* (c-quantile) to distinguish it from a quantile

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regression function which may have been assumed to be linear or estimated nonparametrically, as is common.

A second objective of this paper is to apply the c-quantile idea to assess the form and degree of conditional dependence between foreign exchange rates. Correlation analysis is implicitly based on an assumption of multivariate ellipticity and may give very misleading results if the assumption is incorrect, in particular multivariate Gaussianity implies asymptotic tail area independence unless the correlation is unity and a multivariate t-distribution symmetric tail dependence. An important issue in financial markets is to consider exactly how exchange rates are inter-related in different market conditions, and by using c-quantiles, we show how we can examine the entire conditional distribution at a range of quantile levels rather than just measure the correlation or the degree of limiting dependence which is captured by standard tail area dependency measures. Patton (2006) and Hartmann, Straetmans, and De Vries (2003) have considered the dependence between exchange rates using related but different techniques from those employed in this article. We also consider dynamic dependency both across and within exchange rates and show how the c-quantile method provides an approach different from that considered by Engle and Manganelli (2004) who *assumed* the form of the conditional autoregressive value-at-risk models which they proposed for risk management in their CAViAR framework. (See Bouyé, Gaussel and Salmon, 2001, for a more general approach to assessing dynamic dependence using copulae.) The form of our dynamic c-quantiles follows immediately from the determination of the joint distribution and the copula rather than by assumption. In this way, we are also able to examine the question of market efficiency at all quantiles including the tails of the distributions, instead of simply through a mean regression, by exploiting a natural test for *independence* that follows from the copula.

In the next section, we briefly review regression quantiles as proposed and developed by Koenker and Bassett (1978) and then the concept of copula is defined and the implications for the assessing the forms of dependence between two assets are presented. We then introduce the concept of a copula quantile curve, derive some properties of this c-quantile curve and provide some examples for particular copulae. In the next section, the copula quantile regression model is formally defined and we briefly discuss the estimation issue. Then the application to analysing c-quantile regressions and tail area dependence in foreign exchange markets is presented. A final section offers some conclusions.

2. Regression quantiles

Koenker and Bassett (1978) introduced linear quantile regression in the following way. Let $(y_1, \ldots, y_T)$ be a random sample on Y and $(\mathbf{x}_1, \ldots, \mathbf{x}_T)'$ a random k-vector sample on X.

Definition 1 *The pth quantile regression is any solution to the following problem*:

$$\min_{\boldsymbol{\beta}\in\mathbb{R}^k}\left(\sum_{t\in\mathcal{T}_p} p|y_t - \mathbf{x}_t'\boldsymbol{\beta}| + \sum_{t\in\mathcal{T}_{1-p}}(1-p)|y_t - \mathbf{x}_t'\boldsymbol{\beta}|\right)$$

with $\mathcal{T}_p = \{t\colon y_t \geq \mathbf{x}_t'\boldsymbol{\beta}\}$ and $\mathcal{T}_{1-p}$ its complement. This can be alternatively expressed as:[1]

$$\min_{\boldsymbol{\beta}\in\mathbb{R}^k}\left(\sum_{t=1}^{T}(p - \mathbb{I}_{\{y_t\leq\mathbf{x}_t'\boldsymbol{\beta}\}})(y_t - \mathbf{x}_t'\boldsymbol{\beta})\right). \tag{1}$$

Nonlinearity in quantile regression was developed by Powell (1986) using a censored model and the consistency of nonlinear quantile regression estimation has been investigated by White (1994), Engle and Manganelli (2004) and Kim and White (2003) among others. For a recent overview of quantile regression see Koenker (2005) or Koenker and Hallock (2001). As noted by Buchinsky (1998), quantile regression models have a number of useful properties: (i) with non-Gaussian error terms, quantile regression estimators may be more efficient than least-square estimators, (ii) the entire conditional distribution can be characterized, (iii) different relationships between the regressor and the dependent variable may arise at different quantiles. In this paper, we attempt to resolve one basic issue when using quantile regression, how to specify the functional form of the quantile regression. We achieve this by deriving a conditional distribution for Y given X from the copula which then implies the structural form of the quantile regression. For simplicity, our model is developed for the one regressor case, corresponding to a bivariate copula but it may, in principle, be extended to multiple regressors.

3. Copulae and dependence

We now derive some theoretical properties of copula quantile curves and start by briefly reviewing the definition of a copula function and Sklar's theorem, which ensures the uniqueness of the copula when the margins are continuous. Then, we discuss the concepts of positive quadrant dependence (PQD) and the left tail decreasing (LTD) property and consider how these two concepts are related. These definitions are then used to demonstrate that the concavity (respectively, convexity) of the copula in its first argument induces a positive (respectively, negative) dependence at each quantile level.

DEFINITION 2 *A bivariate copula is a function* $\mathbf{C}: [0,1]^2 \to [0,1]$ *such that*:

(1) $\forall (u,v) \in [0,1]^2$,

$$\begin{cases} \mathbf{C}(u,0) = \mathbf{C}(0,v) = 0 \\ \mathbf{C}(u,1) = u \text{ and } \mathbf{C}(1,v) = v \end{cases} \tag{2}$$

(2) $\forall (u_1, v_1, u_2, v_2) \in [0,1]^4$, $u_1 \leq u_2$ *and* $v_1 \leq v_2$,

$$\mathbf{C}(u_2, v_2) - \mathbf{C}(u_1, v_2) - \mathbf{C}(u_2, v_1) + \mathbf{C}(u_1, v_1) \geq 0. \tag{3}$$

THEOREM 2 (Sklar's theorem) *Let X and Y be two random variables with joint distribution* $\mathbf{F}$. *Then, there exists a unique copula* $\mathbf{C}$ *satisfying*

$$\mathbf{F}(x,y) = \mathbf{C}(F_X(x), F_Y(y)) \tag{4}$$

if F_X and F_Y are continuous and represent the marginal distribution functions of X and Y, respectively.

DEFINITION 3 (Order) *Let* $(\mathbf{C}, \mathbf{D}) \in \mathcal{C}^2$ *with* $\mathcal{C}$ *the set of copulae. One says that* $\mathbf{C}$ *is greater than* $\mathbf{D}$ ($\mathbf{C} \succ \mathbf{D}$ *or* $\mathbf{D} \prec \mathbf{C}$) *if*

$$\forall (u,v) \in [0,1]^2, \quad \mathbf{C}(u,v) \geq \mathbf{D}(u,v).$$

THEOREM 3 (Fréchet bounds) *Let* $\mathbf{C} \in \mathcal{C}$. *Then*,

$$\mathbf{C}^{-} \prec \mathbf{C} \prec \mathbf{C}^{+}$$

where $\mathbf{C}^{-}$ *and* $\mathbf{C}^{+}$ *are such that*

$$\mathbf{C}^{-}(u, v) = \max(u + v - 1, 0)$$
$$\mathbf{C}^{+}(u, v) = \min(u, v).$$

The concept of order for copulae is important as it allows us to rank the dependence between random variables. One interesting copula is the product copula $\mathbf{C}^{\perp}$ – which corresponds to independence – so that $\mathbf{C}^{\perp}(u, v) = uv$ (see Figure 1 below).

DEFINITION 4 (Lehman 1966) *The pair* (X, Y) *is PQD* ($\mathbf{PQD}(X, Y)$) *if*

$$\Pr\{X \leq x, Y \leq y\} \geq \Pr\{X \leq x\}\Pr\{Y \leq y\} \tag{5}$$

In terms of copulae, this definition can be restated $\mathbf{C}^{\perp} \prec \mathbf{C}$.

DEFINITION 5 (Esary and Proschan 1972) *Y is LTD in X* ($\mathbf{LTD}(Y \mid X)$) *if*

$$\forall y, \quad \Pr\{Y \leq y \mid X \leq x\} \text{ is a nonincreasing function of } x. \tag{6}$$

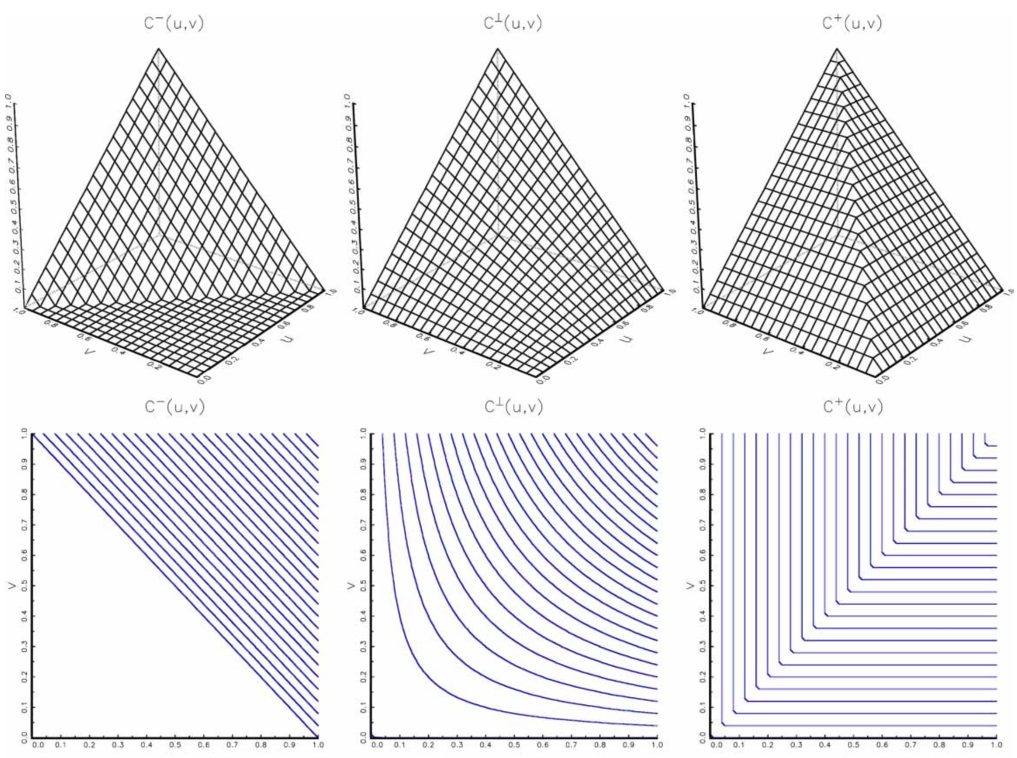

Figure 1. $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$.

This definition can be equivalently expressed using copulae as follows.

THEOREM 4 (Nelsen 1998)

$$\mathbf{LTD}(Y \mid X) \Longleftrightarrow \frac{\mathbf{C}(u,v)}{u} \text{ is nonincreasing in } u$$
$$\Longleftrightarrow \frac{\partial \mathbf{C}(u,v)}{\partial u} \leq \frac{\mathbf{C}(u,v)}{u}. \tag{7}$$

THEOREM 5 *Let $\mathbf{C} \in \mathcal{C}$. The following holds*

$$\text{If } \forall (u,v) \in [0,1]^2, \quad \frac{\partial^2 \mathbf{C}(u,v)}{\partial u^2} \leq 0, \quad \text{then } \mathbf{C}^{\perp} \prec \mathbf{C} \tag{8}$$

$$\text{If } \forall (u,v) \in [0,1]^2, \quad \frac{\partial^2 \mathbf{C}(u,v)}{\partial u^2} \geq 0, \quad \text{then } \mathbf{C} \prec \mathbf{C}^{\perp}. \tag{9}$$

Proof We refer to Nelsen (1998, pp. 151–60), for the proof. The first part is based on the fact that $\partial^2\mathbf{C}(u,v)/\partial u^2 \leq 0 \Rightarrow \mathbf{LTD}(Y \mid X) \Rightarrow \mathbf{PQD}(X,Y)$. □

This theorem tells us that if the copula function is concave in the marginal distribution F_X, then the random variables X and Y are positively related, i.e. their copula value is greater than that given by the independence copula $\mathbf{C}^{\perp}$. Conversely, convexity implies a negative relationship i.e. the copula linking X and Y lies below the independence copula $\mathbf{C}^{\perp}$. For simplicity, we still have not introduced the parameter(s) of the copula function, functions of which measure the degree and different forms of dependence between the variables; let us denote these parameters by $\delta \in \Delta$. Then, through the family of copula functions, we can distinguish three classes.

(1) Copulae that only exhibit negative dependence:

$$\forall \delta \in \Delta, \quad \forall (u,v) \in [0,1]^2, \quad \text{then } \mathbf{C}(u,v;\delta) \prec \mathbf{C}^{\perp}(u,v).$$

(2) Copulae that only exhibit positive dependence:

$$\forall \delta \in \Delta, \quad \forall (u,v) \in [0,1]^2, \quad \text{then } \mathbf{C}^{\perp}(u,v) \prec \mathbf{C}(u,v;\delta).$$

(3) Copulae that exhibit both negative and positive dependence depending on the parameter values:

$$\forall \delta \in \Delta^{-}, \quad \forall (u,v) \in [0,1]^2, \quad \text{then } \mathbf{C}(u,v;\delta) \prec \mathbf{C}^{\perp}(u,v)$$
$$\forall \delta \in \Delta^{+}, \quad \forall (u,v) \in [0,1]^2, \quad \text{then } \mathbf{C}^{\perp}(u,v) \prec \mathbf{C}(u,v;\delta).$$

In the next section, the concept of a quantile curve of Y conditional on X is defined and we derive several results that are directly deduced from the underlying copula properties outlined above.

4. Quantile curves

First, the pth c-quantile curve of y conditional on x or pth c-quantile curve is defined. Second, its main properties are exhibited. Third, the case of radially symmetric variables is studied. Finally, the quantile curves are developed for three special cases: the Kimeldorf and Sampson, Gaussian and Frank copulae.

4.1 *Definitions*

We restrict the study to *monotonic* copula for simplicity. Define the probability distribution of y conditional on x by $p(y|x;\delta)$, where

$$\begin{aligned}
p(y|x;\delta) &= \Pr\{Y \le y \mid X = x\} \\
&= \mathbf{E}(\mathbb{I}_{\{Y\le y\}} \mid X = x) \\
&= \lim_{\varepsilon\to 0} \Pr\{Y \le y \mid x \le X \le x+\varepsilon\} \\
&= \lim_{\varepsilon\to 0} \frac{\mathbf{F}(x+\varepsilon, y;\delta) - \mathbf{F}(x, y;\delta)}{F_X(x+\varepsilon) - F_X(x)} \\
&= \lim_{\varepsilon\to 0} \frac{\mathbf{C}[F_X(x+\varepsilon), F_Y(y);\delta] - \mathbf{C}[F_X(x), F_Y(y);\delta]}{F_X(x+\varepsilon) - F_X(x)} \\
p(y|x;\delta) &= C_1[F_X(x), F_Y(y);\delta], \qquad (10)
\end{aligned}$$

with $C_1(u, v;\delta) = \partial/\partial u C(u, v;\delta)$. Since the distribution functions F_X and F_Y are nondecreasing, $p(y|x;\delta)$ is nondecreasing in y. Using the same argument, $p(y|x;\delta)$ is nondecreasing in x if $C_{11}(u, v;\delta) \le 0$ and nonincreasing in x if $C_{11}(u, v;\delta) \ge 0$, where $C_{11}(u, v;\delta) = \partial^2\mathbf{C}(u, v;\delta)/\partial u^2$.

DEFINITION 6 *For a parametric copula* $\mathbf{C}(\cdot, \cdot;\delta)$ *the pth copula quantile curve of y conditional on x is defined by the following implicit equation*

$$p = C_1[F_X(x), F_Y(y);\delta], \qquad (11)$$

where $\delta \in \Delta$ *the set of parameters.*

Under some conditions,[2] Equation (11) can be expressed as follows in order to capture the relationship between X and Y:

$$y = \mathbf{q}(x, p;\delta), \qquad (12)$$

where $\mathbf{q}(x, p;\delta) = F_Y^{[-1]}(D(F_X(x), p;\delta))$ with D the partial inverse in the second argument of C_1 and $F_Y^{[-1]}$ the pseudo-inverse of F_Y. Note that the relationship (12) can alternatively be written using uniform margins as:

$$v = \mathbf{r}(u, p;\delta), \qquad (13)$$

with $u = F_X(x)$ and $v = F_Y(y)$.

4.2 *Properties*

Two properties are demonstrated. The first tells us that the quantile curve shifts up with the quantile level. The second indicates that the quantile curve has a positive (respectively, negative) slope if the copula function is concave (respectively, convex) in its first argument.

PROPERTY 1 *If* $0 < p_1 \le p_2 < 1$, *then* $\mathbf{q}(x, p_1;\delta) \le \mathbf{q}(x, p_2;\delta)$.

PROPERTY 2 *Let $x_1 \leq x_2$.*
If $\mathbf{C}(u, v)$ is concave in u then $\mathbf{q}(x_1, p; \delta) \leq \mathbf{q}(x_2, p; \delta)$
If $\mathbf{C}(u, v)$ is convex in u then $\mathbf{q}(x_1, p; \delta) \geq \mathbf{q}(x_2, p; \delta)$.

Proof Given the implicit function theorem, y may be expressed as a function of x and p *i.e.* $y = \mathbf{q}(x, p; \delta)$. Let us rewrite Equation (11) as $F(x, p, \mathbf{q}(x, p; \delta)) = 0$. Thus,

$$\frac{\partial F}{\partial x}(x, p, \mathbf{q}(x, p; \delta)) + \frac{\partial F}{\partial y}(x, p, \mathbf{q}(x, p; \delta))\frac{\partial \mathbf{q}}{\partial x}(x, p; \delta) = 0$$
$$\frac{\partial F}{\partial p}(x, p, \mathbf{q}(x, p; \delta)) + \frac{\partial F}{\partial y}(x, p, \mathbf{q}(x, p; \delta))\frac{\partial \mathbf{q}}{\partial p}(x, p; \delta) = 0.$$

Then,

$$\frac{\partial \mathbf{q}}{\partial x}(x, p; \delta) = -\frac{(\partial F/\partial x)(x, p, \mathbf{q}(x, p; \delta))}{(\partial F/\partial y)(x, p, \mathbf{q}(x, p; \delta))}$$
$$\frac{\partial \mathbf{q}}{\partial p}(x, p; \delta) = -\frac{(\partial F/\partial p)(x, p, \mathbf{q}(x, p; \delta))}{(\partial F/\partial x)(x, p, \mathbf{q}(x, p; \delta))}.$$

Just note that $F(x, p, y) = C_{1.}[F_X(x), F_Y(y); \delta] - p$, it follows that

$$\frac{\partial \mathbf{q}}{\partial x}(x, p; \delta) = -\frac{f_X(x)C_{2.}[F_X(x), F_Y(y); \delta]}{f_Y(y)C_{11}[F_X(x), F_Y(y); \delta]}$$
$$\frac{\partial \mathbf{q}}{\partial p}(x, p; \delta) = \frac{1}{f_Y(y)C_{11}[F_X(x), F_Y(y); \delta]}. \tag{14}$$

As $\forall (u, v) \in [0, 1]^2$, $C_{11}[u, v; \delta] \geq 0$, $f_X(x) \geq 0$ and $f_Y(y) \geq 0$, this completes the proof. □

4.3 *Symmetric case*

An interesting case concerns the radial symmetry of X and Y. Indeed, in this case, a remarkable relationship exists between the pth quantile curve and the $(1 - p)$th quantile curve. First, the definition of radial symmetry is given. Then, a theorem is stated and a corollary that informs us about the slopes of the quantile curves is provided.

DEFINITION 7 *Two random variables X and Y are radially symmetric about (a, b) if*

$$\Pr\{X \leq x - a, Y \leq y - b\} = \Pr\{X \geq x + a, Y \geq y + b\}. \tag{15}$$

THEOREM 6 (Nelsen 1998) *Let X and Y be, respectively, symmetric about a and b. They are radially symmetric about (a, b) iff their copula $\mathbf{C}$ satisfies*:

$$\mathbf{C}(u, v) = u + v - 1 + \mathbf{C}(1 - u, 1 - v). \tag{16}$$

COROLLARY 7 *If the copula $\mathbf{C}$ satisfies Equation* (11), *then*

$$C_{1.}(u, v; \delta) = 1 - C_{1.}(1 - u, 1 - v; \delta)$$
$$C_{2.}(u, v; \delta) = C_{2.}(1 - u, 1 - v; \delta)$$
$$C_{11}(u, v; \delta) = C_{11}(1 - u, 1 - v; \delta).$$

THEOREM 8 (Radial symmetry and copula quantile curves) *If two random variables X and Y are radially symmetric about* (a, b), *then*

$$\mathbf{q}(a - x, p; \delta) + \mathbf{q}(a + x, 1 - p; \delta) = 2b. \tag{17}$$

Proof From Equation (15),

$$\Pr\{Y \leq y - b \mid X \leq x - a\} = \Pr\{Y \geq y + b \mid X \geq x + a\}.$$

In terms of copula,

$$C_{1.}[F_X(a - x), F_Y(b - y); \delta] = 1 - C_{1.}[F_X(a + x), F_Y(b + y); \delta]$$
$$p(a - x, b - y) = 1 - p(a + x, b + y)$$

Then, for $p(a - x, b - y) = p$,

$$b - y = \mathbf{q}(a - x, p; \delta)$$
$$b + y = \mathbf{q}(a + x, 1 - p; \delta)$$

and the proof follows. □

Note that a direct implication of this theorem is $\mathbf{q}(a, 1/2; \delta) = b$.

COROLLARY 9 *If two random variables X and Y are radially symmetric about* (a, b), *then*

$$\frac{\partial \mathbf{q}}{\partial x}(a - x, p; \delta) = \frac{\partial \mathbf{q}}{\partial x}(a + x, 1 - p; \delta). \tag{18}$$

4.4 *Examples*

We first describe a case for the Kimeldorf and Sampson copula where the copula quantiles can be derived analytically. We then describe how to develop c-quantiles for the general class of Archimedean copulae and two specific Archimedean copulae that we use in the empirical analysis below: the Joe–Clayton copula (BB7 in Joe 1997), which was used by Patton (2006), and BB3. We then study two copulae that allow both positive and negative slopes for the quantile curves, depending on the value of their dependence parameter. These are the Gaussian copula where the dependence pattern is measured by correlation but where the marginal distributions may be non-Gaussian. We then show that we have to be careful when selecting copula since some copulae, such as the Frank copula, may not allow us to adequately capture the full range of behaviour in the distribution of the dependent variable Y.

4.4.1 Kimeldorf and Sampson copula

Consider the copula given by

$$C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} \text{ for } \theta > 0,$$

we then have

$$C_1(v|u) = \frac{\partial C(u,v)}{\partial u}$$
$$= -\frac{1}{\theta}(u^{-\theta} + v^{-\theta} - 1)^{-(1+\theta)/\theta}(-\theta u^{-(1+\theta)})$$
$$= (1 + u^{\theta}(v^{-\theta} - 1))^{-(1+\theta)/\theta}$$

solving $p = C_1(v|u)$ for v gives

$$C_1^{-1}(v|u) = v = (p^{(-\theta/1+\theta)} - 1)u^{-\theta} + 1)^{-1/\theta},$$

which provides us with the c-quantiles relating v and u for different values of p. Using the empirical distribution functions for $u = F_X(x)$ and $v = F_Y(y)$, we can find explicit expressions for the conditional c-quantiles for the variable Y conditional on X

$$y = F_Y^{-1}((p^{(-\theta/1+\theta)} - 1)F_X(x)^{-\theta} + 1)^{-1/\theta}).$$

4.4.2 *Archimedean copulae*

4.4.2.1 General case. An Archimedean copula is defined as follows:

$$C(u,v) = \phi^{-1}[\phi(u) + \phi(v)] \tag{19}$$

with ϕ a continuous and strictly decreasing function from [0, 1] to [0, ∞] such that $\phi(1) = 0$. ϕ is often called the generator function. From $p = \partial/\partial u C(u,v)$, we obtain

$$p = \frac{\phi'(u)}{\phi'(C(u,v))}$$
$$p = \frac{\phi'_{\delta}(u)}{\phi'(\phi^{-1}[\phi(u) + \phi(v)])} \tag{20}$$

and the quantile regression curve for Archimedean copulae can in general be deduced as

$$v = \mathbf{r}(u, p; \delta)$$
$$v = \phi^{-1}\left[\phi\left(\phi'^{-1}\left(\frac{1}{p}\phi'(u)\right)\right) - \phi(u)\right].$$

Introducing $u = F_X(x)$ and $v = F_Y(y)$, the equation for the c-quantile above becomes

$$y = F_Y^{-1}\left(\phi^{-1}\left[\phi\left(\phi'^{-1}\left(\frac{1}{p}\phi'(F_X(x))\right)\right) - \phi(F_X(x))\right]\right).$$

4.4.3 *Two specific Archimedean copulae*

4.4.3.1 Joe–clayton (BB7). For the copula defined by

$$C_{\delta,\theta}(u,v) = 1 - (1 - [(1 - (1-u)^{\theta})^{-\delta} + (1 - (1-v)^{\theta})^{-\delta} - 1]^{-1/\delta})^{1/\theta} \tag{21}$$

with $\theta \geq 1$ and $\delta > 0$, see Joe (1997, p. 153). This two-parameter copula is Archimedean as

$$C_{\delta,\theta}(u,v) = \phi_{\delta,\theta}^{-1}[\phi_{\delta,\theta}(u) + \phi_{\delta,\theta}(v)]$$

with

$$\phi_{\delta,\theta}(s) = [1-(1-s)^\theta]^{-\delta} - 1$$
$$\phi^{-1}_{\delta,\theta}(s) = 1-[1-(1+s)^{-1/\delta}]^{1/\theta}$$
$$\phi'_{\delta,\theta}(s) = -[1-(1-s)^\theta]^{-\delta-1}\delta[-(1-s)^\theta\theta/(-1+s)] \quad (22)$$

It only allows positive dependence and we can see that

$$\lim_{\delta\to\infty} C_{\delta,\theta}(u,v) = C^+(u,v)$$
$$\lim_{\theta\to\infty} C_{\delta,\theta}(u,v) = C^+(u,v).$$

An important property is that each parameter, respectively, measures lower (δ) and upper (θ) tail dependence as we show below. Moreover this copula encompasses two copulae sub-families as for $\theta = 1$ one obtains the Kimeldorf and Sampson (1975) copula:

$$C_\delta(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta},$$

and for $\delta \to 0$ the Joe (1993) copula:

$$C_\theta(u,v) = 1-((1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta)^{1/\theta}.$$

4.4.3.2 BB3. For the BB3 copula defined below (Joe 1997),

$$C_{\delta,\theta}(u,v) = \exp(1-[\delta^{-1}\ln(\exp(\delta\tilde{u}^\theta) + \exp(\delta\tilde{v}^\theta) - 1)]^{1/\theta}) \quad (23)$$

with $\theta \geq 1$ and $\delta > 0$. This copula is archimedean as

$$C_{\delta,\theta}(u,v) = \phi^{-1}_{\delta,\theta}[\phi_{\delta,\theta}(u) + \phi_{\delta,\theta}(v)]$$

with

$$\phi^{-1}_{\delta,\theta}(s) = \exp(-[\delta^{-1}\ln(1+s)]^{1/\theta}) \quad (24)$$

Again this copula allows us to model positive dependence and

$$\lim_{\delta\to\infty} C_{\delta,\theta}(u,v) = C^+(u,v)$$
$$\lim_{\theta\to\infty} C_{\delta,\theta}(u,v) = C^+(u,v).$$

The lower and upper tail area dependence measures are given by

$$\lambda_L = \begin{cases} 2^{-1/\delta} \text{ if } \theta = 1 \\ 1 \text{ if } \theta > 1 \end{cases}$$
$$\lambda_U = 2 - 2^{1/\theta}. \quad (25)$$

Again each parameter, respectively, measures lower (δ) and upper (θ) tail dependence.

4.4.4 Gaussian copula

The bivariate copula in this case is written as:

$$\mathbf{C}(u, v; \rho) = \Phi_2(\Phi^{[-1]}(u), \Phi^{[-1]}(v); \rho) \tag{26}$$

with Φ_2 the bivariate Gaussian distribution and Φ the univariate distribution.

$$p = \Phi\left(\frac{\Phi^{[-1]}(v) - \rho\Phi^{[-1]}(u)}{\sqrt{1-\rho^2}}\right)$$

or equivalently solving for v we find the pth c-quantile curve (see Figure 2) to be

$$v = \mathbf{r}(u, p; \rho) = \Phi\left(\rho\Phi^{[-1]}(u) + \sqrt{1-\rho^2}\Phi^{[-1]}(p)\right).$$

The slope of the pth quantile curve is given by (Figure 2):

$$\frac{\partial \mathbf{r}(u, p; \rho)}{\partial u} = \rho \frac{\phi\left(\rho\Phi^{[-1]}(u)\sqrt{1-\rho^2}\Phi^{[-1]}(p)\right)}{\phi(\Phi^{[-1]}(u))}.$$

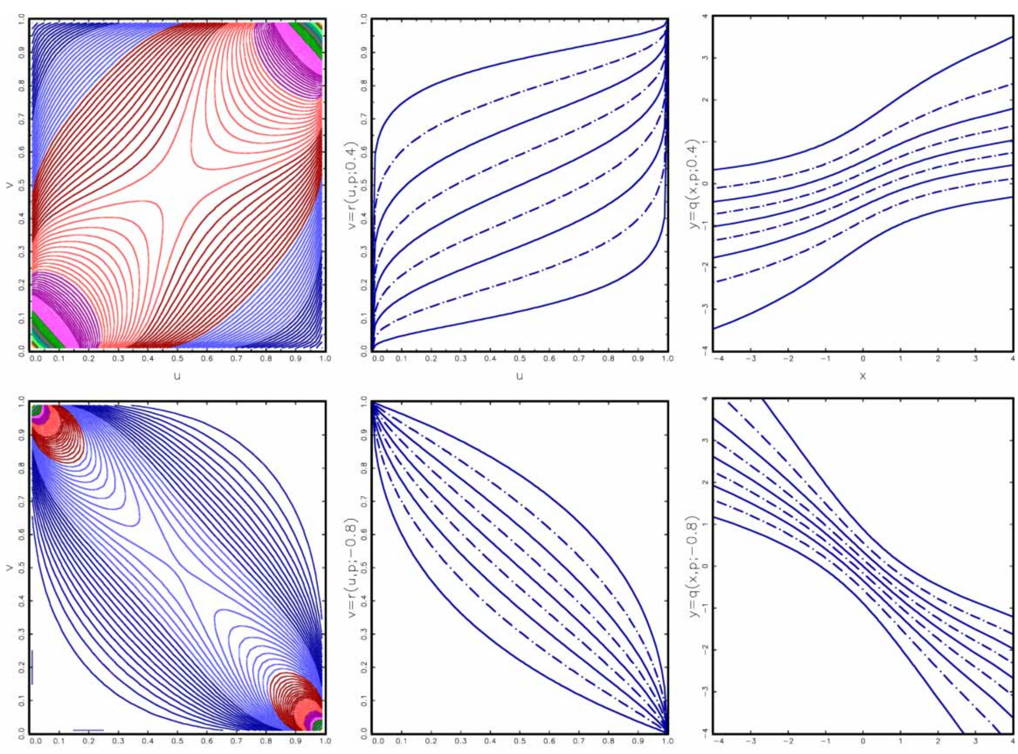

Figure 2. Gaussian copula densities, pth copula quantile curves (for $p = 0.1, 0.2, \ldots, 0.9$) for (u, v) and (x, y) under the hypothesis of Student's margins ($\nu = 3$) for $\rho = 0.4$ (upper plots) and $\rho = -0.8$ (lower plots).

A positive correlation is characterized by a positive slope and conversely for a negative correlation. Moreover,

$$\frac{\partial \mathbf{r}(u, p; \rho)}{\partial p} = \sqrt{1-\rho^2}\frac{\phi\left(\rho\Phi^{[-1]}(u) + \sqrt{1-\rho^2}\Phi^{[-1]}(p)\right)}{\phi(\Phi^{[-1]}(u))},$$

which is always positive. Then, the higher the p, the higher the quantile curve. The relationship between y and x for the pth quantile is:

$$y = F_Y^{[-1]}\left[\Phi\left(\rho\Phi^{[-1]}(F_X(x)) + \sqrt{1-\rho^2}\Phi^{[-1]}(p)\right)\right]. \tag{27}$$

Let assume that X and Y are jointly bivariate Gaussian with $\mu_X = \mathbf{E}[X]$, $\mu_Y = \mathbf{E}[Y]$, $\sigma_X^2 = \mathrm{Var}[X]$, $\sigma_Y^2 = \mathrm{Var}[Y]$ and $\rho = \mathrm{Corr}[X, Y]$. Then, given in Equation (27), the relationship becomes linear and we have

$$y = \mathbf{q}(x_t, p; \rho) = a + bx$$

with slope and intercept values determined by the quantile level;

$$a = \mu_Y + \sigma_Y\sqrt{1-\rho^2}\Phi^{[-1]}(p) - \rho\frac{\sigma_Y}{\sigma_x}\mu_X$$

$$b = \rho\frac{\sigma_Y}{\sigma_x}$$

4.4.5 Frank copula

This copula is given by

$$\mathbf{C}(u, v; \delta) = -\frac{1}{\delta}\ln\left(1 + \frac{(\mathrm{e}^{-\delta u}-1)(\mathrm{e}^{-\delta v}-1)}{\mathrm{e}^{-\delta}-1}\right). \tag{28}$$

By computing its first derivative with respect to u, one obtains the pth copula quantile curve, $p = \mathbf{C}_{1.}(u, v; \delta)$ as

$$p = \mathrm{e}^{-\delta u}((1-\mathrm{e}^{-\delta})(1-\mathrm{e}^{-\delta v})^{-1} - (1-\mathrm{e}^{-\delta u}))^{-1}$$

or equivalently,

$$v = -\frac{1}{\delta}\ln(1 - (1-\mathrm{e}^{-\delta})[1 + \mathrm{e}^{-\delta u}(p^{-1}-1)]^{-1}).$$

From the definition of the uniform distribution, one obtains the nonlinear relationship between x and y for the pth quantile as:

$$y = F_Y^{[-1]}\left[-\frac{1}{\delta}\ln(1 - (1-\mathrm{e}^{-\delta})[1 + \mathrm{e}^{-\delta F_X(x)}(p^{-1}-1)]^{-1})\right]. \tag{29}$$

We can see that the Frank copula might not always be a good choice as shown in Figure 3 since the full range of potential values for the variables may not be captured. So for $u \in [0, 1]$,

$$-\frac{1}{\delta}\ln(1-(1-\mathrm{e}^{-\delta})p) \le \mathbf{r}(u, p; \delta) \le -\frac{1}{\delta}\ln\left(\frac{1-\mathrm{e}^{-\delta}}{1+\mathrm{e}^{-\delta}(p^{-1}-1)}\right) \text{ for } \delta > 0$$

and

$$-\frac{1}{\delta}\ln(1-(1-\mathrm{e}^{-\delta})p) \ge \mathbf{r}(u, p; \delta) \ge -\frac{1}{\delta}\ln\left(\frac{1-\mathrm{e}^{-\delta}}{1+\mathrm{e}^{-\delta}(p^{-1}-1)}\right) \text{ for } \delta < 0.$$

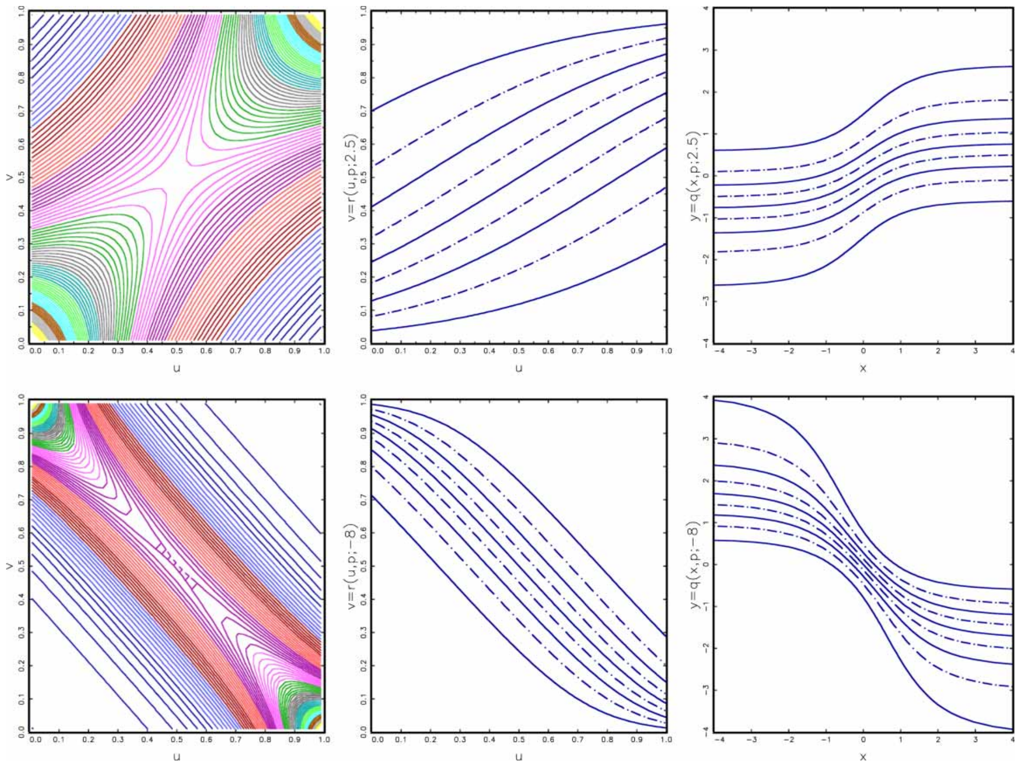

Figure 3. Frank copula densities, copula pth quantile curves (for $p = 0.1, 0.2, \ldots, 0.9$) for (u, v) and (x, y) under the hypothesis of Student's margins ($\nu = 3$) for $\delta = 2.5$ (upper plots) and $\delta = -8$ (lower plots).

5. Copula quantile regression and tail area dependency

Given the development above the concept of a copula quantile regression can be seen to be just a special case of nonlinear quantile regression. Let $(y_1, \ldots, y_T)$ be a random sample on Y and $(x_1, \ldots, x_T)$ a random k-vector sample on X.

DEFINITION 8 *The pth copula quantile regression* $\mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta})$ *is a solution to the following problem*:

$$\min_{\boldsymbol{\delta}} \left(\sum_{t \in \mathcal{T}_p} p |y_t - \mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta})| + \sum_{t \in \mathcal{T}_{1-p}} (1-p) |y_t - \mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta}).| \right) \tag{30}$$

with $\mathcal{T}_p = \{t : y_t \geq \mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta})\}$ *and* $\mathcal{T}_{1-p}$ *its complement. This can be expressed alternatively as:*

$$\min_{\boldsymbol{\delta}} \left(\sum_{t=1}^{T} (p - \mathbb{I}_{\{y_t \leq \mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta})\}})(y_t - \mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta})) \right). \tag{31}$$

This definition indicates that the estimate of the dependence parameter δ is provided by an L^1 norm estimator. This general problem has been investigated by Koenker and Park (1996) who proposed an algorithm for problems with response functions that are nonlinear in parameters and we refer to their paper for a detailed discussion of the development of an interior point algorithm to solve the estimation problem. The main idea is to solve the nonlinear L^1 problem by splitting it into a succession of linear L^1 problems.

It might be surprising that the probability level p appears in Equation (30) as an argument of the function $\mathbf{q}$ itself. This is simply because we have adopted a top-down strategy in our modelling by first specifying the joint distribution and then deriving the implied quantile function. By postulating given margins for X and Y and their copula, we derive a specific parametric functional for $\mathbf{q}(\mathbf{x}_t, p; \boldsymbol{\delta})$. In fact, the probability level is implicit in the original quantile regression definition of Koenker and Bassett (1978). In the applications below, we use nonparametric estimates of the empirical marginal distribution functions but estimate the parameters of the copula quantile parameters, $\boldsymbol{\delta}$, as described above. Conditions for the consistency of this semiparametric approach to the estimation of copula-based time-series models have been discussed in Chen and Fan (2006).

The c-quantile approach, developed above, enables us to examine the dependency between assets at any given quantile, including extreme quantiles and we can now compare this approach with the standard asymptotic tail area dependency measures. We may in fact not often be interested in dependency in the far extremes where highly infrequent but potentially disasterous joint loss may occur and we may be more interested in the more frequent dependency where large but not extreme loss can arise, and in this latter case, the c-quantile approach should provide a better measure of association between the assets.

5.1 *Tail area dependency*

Tail dependence measures, for both the upper tail, λ_{U}, and lower tail, λ_{L}, have been developed, and discussed, for instance in Joe (1997), Mari and Kotz (2001) and Coles, Heffernan and Tawn (1999). Upper tail independence is normally thought to be shown by $\lambda_{\mathrm{U}} = 0$ and a value of $\lambda_{\mathrm{U}} \in (0, 1]$ indicates the degree of upper tail dependence with lower tail dependence $\lambda_{\mathrm{L}} \in (0, 1]$ similarly defined.

For two random variables, (X, Y) with marginal distributions $F_1(X)$ and $F_2(Y)$, λ_{U} and λ_{L} are linked to the asymptotic behaviour of the copula in the left and right tails. So for the lower tail index, we have

$$\lambda_{\mathrm{L}} = \lim_{\alpha \searrow 0} \frac{C(\alpha, \alpha)}{\alpha} \tag{32}$$

$$= \lim_{\alpha \searrow 0} (\Pr\{F_2(Y) \leq \alpha | F_1(X) \leq \alpha\})$$

$$= \lim_{\alpha \searrow 0} (\Pr\{Y \leq F_2^{-1}(\alpha) | X \leq F_1^{-1}(\alpha)\}) \tag{33}$$

and

$$\lambda_{\mathrm{U}} = \lim_{\alpha \nearrow 1} \Pr\{Y > F_2^{-1}(\alpha) | X > F_1^{-1}(\alpha)\}$$

$$= \lim_{\alpha \nearrow 1} \frac{\Pr\{Y > F_2^{-1}(\alpha), X > F_1^{-1}(\alpha)\}}{\Pr\{X > F_1^{-1}(\alpha)\}}$$

$$= \lim_{\alpha \nearrow 1} \frac{\bar{C}(\alpha, \alpha)}{1-\alpha}$$
$$= \lim_{\alpha \nearrow 1} \frac{1 - 2\alpha + C(\alpha, \alpha)}{1-\alpha} \quad (34)$$

Given the survival copula of two random variables with copula $C(\cdot, \cdot)$ is given by

$$\hat{C}(u, v) = u + v - 1 + C(1-u, 1-v),$$

and the joint survival function for two uniform random variables with distribution function given by $C(u, v)$ is given by

$$\bar{C}(u, v) = 1 - u - v + C(u, v) = \hat{C}(1-u, 1-v);$$

hence, it follows that

$$\lim_{\alpha \nearrow 1} \frac{\bar{C}(\alpha, \alpha)}{1-\alpha} = \lim_{\alpha \nearrow 1} \frac{\hat{C}(1-\alpha, 1-\alpha)}{(1-\alpha)} = \lim_{\alpha \searrow 0} \frac{\hat{C}(\alpha, \alpha)}{\alpha}. \quad (35)$$

Which implies that the coefficient of upper tail dependence of $C(\cdot, \cdot)$ is the coefficient of lower tail dependence of $\hat{C}(\cdot, \cdot)$.

A major difficulty with interpreting asymptotic tail area dependency, however, is that independence in the sense of *the factorization of the bivariate distribution* in the tails implies $\lambda_U = 0$ but $\lambda_U = 0$ does not imply factorization and hence independence. There may still be dependence in the tails even though there is asymptotic independence. An additional condition must be used to ensure factorization; Ledford and Tawn (1998), for instance, show that we also need to satisfy $\bar{\lambda} = 0$ as a necessary condition, where

$$\bar{\lambda} = \lim_{\alpha \nearrow 1} \frac{2 \log \Pr\{X > F_X^{-1}(\alpha)\}}{\log \Pr\{X > F_X^{-1}(\alpha), Y > F_Y^{-1}(\alpha)\}} - 1$$
$$= \lim_{\alpha \nearrow 1} \frac{2 \log(1-\alpha)}{\log[1 - 2\alpha + C(\alpha, \alpha)]} - 1$$
$$= \lim_{\alpha \nearrow 1} \frac{2 \log(1-\alpha)}{\log \bar{C}(\alpha, \alpha)} - 1$$

if $\bar{\lambda} > 0$ large values occur simultaneously more frequently than if they were independent and if $\bar{\lambda} < 0$ simultaneous large movements occur less frequently than under independence. $\bar{\lambda} = 1$ if and only if $\lambda_U > 0$ while it takes values in $[-1, 1)$ when $\lambda_U = 0$ which enables us to quantify the strength of dependence in the tail. Values of $\bar{\lambda} > 0$, $\bar{\lambda} = 0$, $\bar{\lambda} < 0$ loosely correspond to when the variables are positively associated in the extremes, exactly independent and negatively associated.

The two indices $(\lambda_U, \bar{\lambda})$ are then used to measure extreme upper tail dependence:

(1) $(\lambda_U > 0, \bar{\lambda} = 1)$ indicates asymptotic dependence and λ_U measures the degree of upper tail dependence or
(2) $(\lambda_U = 0, \bar{\lambda} < 1)$ indicates asymptotic independence and $\bar{\lambda}$ measures the strength of dependence in the tail.

5.1.1 Parametric and nonparametric estimation of tail dependency

The standard lower and upper tail dependence measures for Archimedean copulae are defined in general by

$$\lambda_L = \lim_{\alpha \to 1^-} \frac{1 - 2\alpha + \phi^{-1}(2\phi(\alpha))}{1-\alpha}$$
$$\lambda_U = \lim_{\alpha \to 0^+} \frac{\phi^{-1}(2\phi(\alpha))}{\alpha} \tag{36}$$

and for the Joe–Clayton copula specifically are given by

$$\lambda_L = 2^{-1/\delta}$$
$$\lambda_U = 2 - 2^{1/\theta}; \tag{37}$$

hence MLE estimates of the parameters of the copula provide direct parametric estimates of the tail area dependency measures and we shall employ these formula below.

Alternatively, we can use an empirical or nonparametric copula $C_n(1/n, j/n) = 1/n \sum_{i=1}^{n} \mathcal{I}(X_i \leq X_{(i)}, Y_i \leq Y_{(i)})$ to estimate tail area dependency given the order statistics $X_{(1)} \leq X_{(2)} \leq \cdots X_{(n)}$ and $Y_{(1)} \leq Y_{(2)} \leq \cdots Y_{(n)}$. Since we have

$$\lambda_L = \lim_{\alpha \searrow 0} \frac{C(\alpha,\alpha)}{\alpha},$$

which implies

$$C(\alpha,\alpha) = \lambda_L \alpha + o(\alpha) \tag{38}$$

for $0 \leq \alpha \leq 1$, where $o(\alpha)/\alpha \to 0$ as $\alpha \to 0$. A natural estimator of λ_L is given by the derivative which is approximated by the secant

$$\hat{\lambda}_L^1 = \left(\frac{k}{n}\right)^{-1} \hat{C}_n\left(\frac{k}{n}, \frac{k}{n}\right).$$

Alternatively a least squares estimator can be applied to Equation (38) to give a second estimator

$$\hat{\lambda}_L^2 = \left(\sum_{i=1}^{k}\left(\frac{i}{n}\right)^2\right)^{-1} \sum_{i=1}^{k}\left(\frac{i}{n}.\hat{C}_n\left(\frac{i}{n}, \frac{i}{n}\right)\right). \tag{39}$$

A third estimator is given by recognizing that the copula $C(u,v)$ can be approximated by the mixture of the comonotonicity and independence copulae, M and Π, giving

$$C(\alpha,\alpha) = \lambda_L \alpha + (1-\lambda_L)\alpha^2.$$

If we rewrite this as

$$C(\alpha,\alpha) - \alpha^2 = \lambda_L(\alpha - \alpha^2)$$

and again apply least squares to this expression, we get a third estimator

$$\hat{\lambda}_L^3 = \frac{\sum_{i=1}^{k}(\hat{C}_n(i/n, i/n) - (i/n)^2)(i/n - (i/n)^2)}{\sum_{i=1}^{k}(i/n - (i/n)^2)^2}.$$

As shown by Dobrić and Schmidt (2005), each of these are consistent estimators for λ_L provided k, the number of observations in the lower tail, is chosen so that $k \approx \sqrt{n}$. Our own experimentation with these estimators suggests that $\hat{\lambda}_L^2$ is the most reliable, with $\hat{\lambda}_L^3$ giving values that at times fall outside the feasible range $(0, 1)$. Dobrić and Schmidt provide some Monte Carlo evidence on the relative merits of each estimator depending on the sample size and the true degree of dependence. We use $\hat{\lambda}_L^2$ below to calculate both lower and upper tail dependence using the relationship in Equation (35).

6. Measuring dependence in FX markets

We now turn to apply the methods discussed above and to examine the dependency between three exchange rates, both in the extremes and at a range of quantiles describing the conditional distributions. We start by considering the static relationship between the dollar–yen, dollar–sterling and dollar–DM rates using 522 *weekly* returns from August 1992 to August 2002; the rates themselves are shown in the Figure 4. below. We then turn to consider the dynamic dependence of conditional quantiles both within and between these rates. All three exchange rates fail univariate normality tests with excess kurtosis and a positive skew except for the sterling–dollar rate which shows a negative skew over the sample period (see Figures 5, 6, 7 and Figures 14, 15, 16 for non parametric quantile regressions). For comparison purposes, we follow Patton (2006) and start by imposing a Gaussian copula to combine these non-Gaussian marginals and then examine the sensitivity of our conclusions by using the Joe–Clayton (BB7) copula. The Gaussian copula is examined first simply because multivariate Gaussianity is a standard assumption in practice and also because we know that the Gaussian Copula implies asymptotic independence and hence it provides a useful basis for a comparison between quantile dependence and the tail area dependence measures.

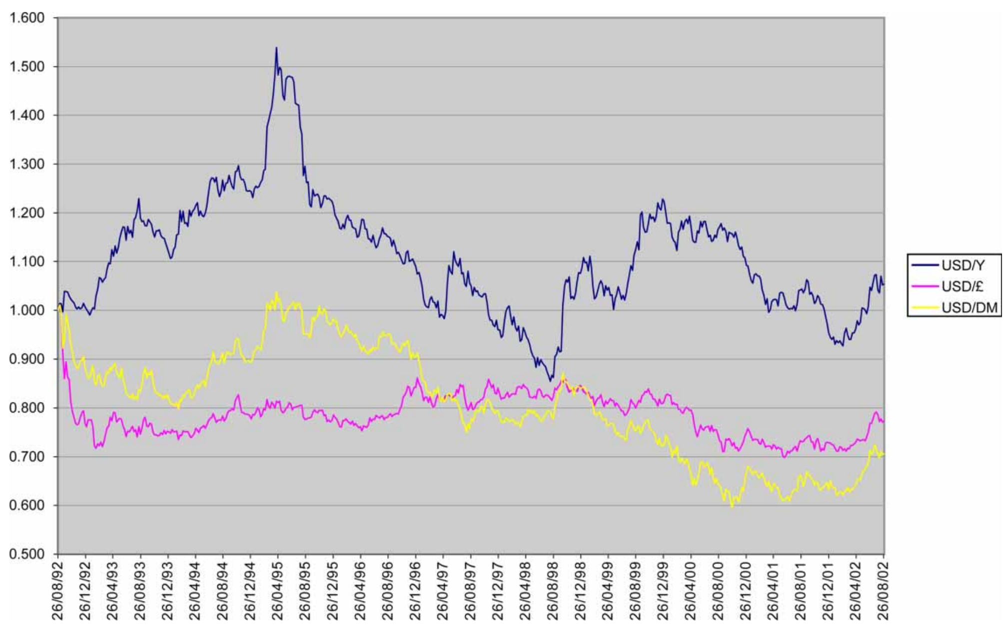

Figure 4. Exchange rates.

We compute the nonlinear quantile regression estimates of $\hat{\rho}(p)$ such that:

$$\hat{\rho}(p) = \arg\min \left(\sum_{t=1}^{T} \left(p - \mathbb{I}_{\{S_{1t} \leq \mathbf{q}(S_{2t}, p; \rho, \hat{\theta}_1, \hat{\theta}_2)\}} \right) \left(S_{1t} - \mathbf{q} \left(S_{2t}, p; \rho, \hat{\theta}_1, \hat{\theta}_2 \right) \right) \right). \quad (40)$$

Assuming a Gaussian copula, the relationship between any two exchange rates S_1 and S_2 at the pth-quantile is[3]:

$$S_1 = \hat{F}_1^{[-1]} \left[\Phi(\hat{\rho}(p)\Phi^{[-1]}(\hat{F}_2(S_2)) + \sqrt{1 - \hat{\rho}^2(p)}\Phi^{[-1]}(p)) \right], \quad (41)$$

with $\hat{F}_1$ and $\hat{F}_2$ the *empirical* marginal distribution functions for the two exchange rates. The estimates of the copula parameter (which in this case is just the correlation coefficient) at each quantile level $\hat{\rho}(p)$, expressed in percentage terms, are reported in Tables 1 and 2 together with their estimated standard deviations. The mean regression results are also reported for information. The lower the p, the higher the quantile regression curve.

The results in Tables 1 and 2 indicate significant (ie. nonzero) dependence using standard inference procedures at all quantile levels and for all exchange rates using the Gaussian copula.[4] There is a relatively low degree of association indicated between the yen:dollar and the sterling:dollar rates and a much higher association indicated at all quantile levels for the dollar:sterling and dollar:DM rates. A fairly symmetric degree of dependence is indicated as we range from the 5% quantile to the 95% quantile with relatively minor differences from the mean regression results. We find the same qualitative conclusions in these two cases when we reverse the causality in Table 2. What is striking, however, are the results for yen:dollar and DM:dollar, where we can see a clear asymmetric structure in the dependency between the lower quantiles and the upper quantiles with much stronger dependency being shown in the lower quantiles when the yen is the dependent variable (and vice versa in the upper quantiles when the DM is the dependent variable). Use of the mean or median regression in this case could give a substantially misleading idea of the

Table 1. c-Quantile regression estimates based on a Gaussian copula (with standard errors).

S_1 on S_2	USD/Y on USD/£	USD/Y on USD/DM	USD/£ on USD/DM
5%	14.2 (5.4)	37.7 (3.5)	49.1 (4.6)
10%	16.5 (4.7)	31.9 (4.2)	57.2 (4.0)
50%	20.2 (3.8)	32.9 (4.0)	72.0 (3.1)
90%	14.1 (5.5)	28.5 (4.7)	63.2 (3.6)
95%	13.2 (5.9)	23.3 (5.7)	55.8 (4.1)
Mean regression	18.3 (4.2)	32.0 (4.2)	65.2 (3.5)

Table 2. Reverse c-quantile regression estimates based on a Gaussian copula.

S_1 on S_2	USD/£ on USD/Y	USD/DM on USD/Y	USD/DM on USD/£
5%	14.4 (5.9)	21.4 (6.2)	51.2 (4.0)
10%	17.5 (4.9)	20.1 (6.6)	57.7 (3.6)
50%	20.5 (4.1)	33.4 (4.0)	64.3 (3.2)
90%	22.8 (3.7)	37.1 (3.6)	66.1 (3.1)
95%	16.9 (5.0)	34.3 (3.9)	51.2 (4.0)
Mean regression	19.2 (4.4)	32.0 (4.2)	62.1 (3.3)

relative joint risks. These results clearly show that there is still a fair degree of quantile dependence at both the upper and lower tails even though we know that the standard tail dependence measures would indicate independence since we are using the Gaussian copula. We have also found a fair degree of symmetry in the dependency between the rates using copula quantiles except in the case of yen:dollar and DM:dollar where we find asymmetry; results that are broadly in line with Patton (2006). Different information is provided by the quantile dependence measures at fairly extreme quantiles than that shown by the implied (asymptotic) tail area dependence measure.

We next compare these Gaussian copula results with those from using the Joe–Clayton copula[5] in Tables 3 (returns) and 4 (levels) where the stars in the following tables indicate significance at the 95% level from the value of one for θ (upper tail dependency) and zero for δ. (lower tail dependency). We can see the same indication of upper tail dependence in the yen:DM dollar rates in levels and sterling:DM dollar rates in the upper tail in returns but not in levels in contrast to the Gaussian copula results. Some lower tail dependence is found for the yen:sterling dollar rates and sterling:DM dollar rates in levels and more strongly in the sterling:DM in returns. Otherwise we find little or no dependence at all with $\hat{\theta}(p)$ being approximately 1 and $\hat{\delta}(p)$ not significantly different from 0 for most quantile levels. The obvious advantage from using the Joe–Clayton copula is that we can separate the dependence parameters θ and δ with their distinct interpretations from the correlation which describes the entire dependence structure with the Gaussian copula.

Next we compute the upper and lower tail indices for the returns of the three exchange rates using both the nonparametric estimator λ_L^2 discussed above and then the parametric estimates using these estimated copula parameters. The nonparametric estimates are shown in Table 5 and

Table 3. Joe–clayton c-quantile regression estimates: returns.

	r_t					
	USD/Y on USD/£		USD/Y on USD/DM		USD/£ on USD/DM	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.07	0.00	1.17*	0.00	1.42*	0.19*
10%	1.07	0.00	1.17*	0.00	1.39*	0.21*
50%	1.06	0.03	1.13	0.09	1.21*	0.37*
90%	1.05	0.08	1.10	0.21	1.07	0.53*
95%	1.04	0.09	1.09	0.23	1.06	0.55*

Table 4. Joe–Clayton c-quantile regression estimates: levels.

	S_t					
	USD/Y on USD/£		USD/Y on USD/DM		USD/£ on USD/DM	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.01	0.67*	1.39*	0.00	1.04	0.44*
10%	1.02	0.54*	1.39*	0.00	1.03	0.44*
50%	1.00	0.00	1.37*	0.00	1.00	0.35*
90%	1.00	0.00	1.25*	0.17	1.00	0.15
95%	1.00	0.00	1.24*	0.20	1.00	0.11

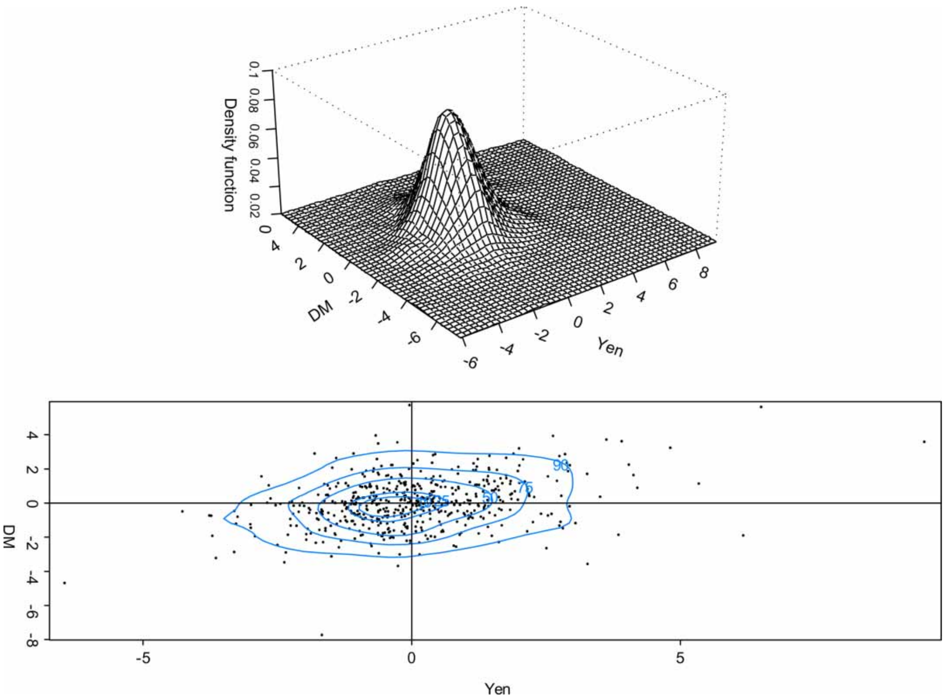

Figure 5. Empirical yen/DM bivariate return distribution.

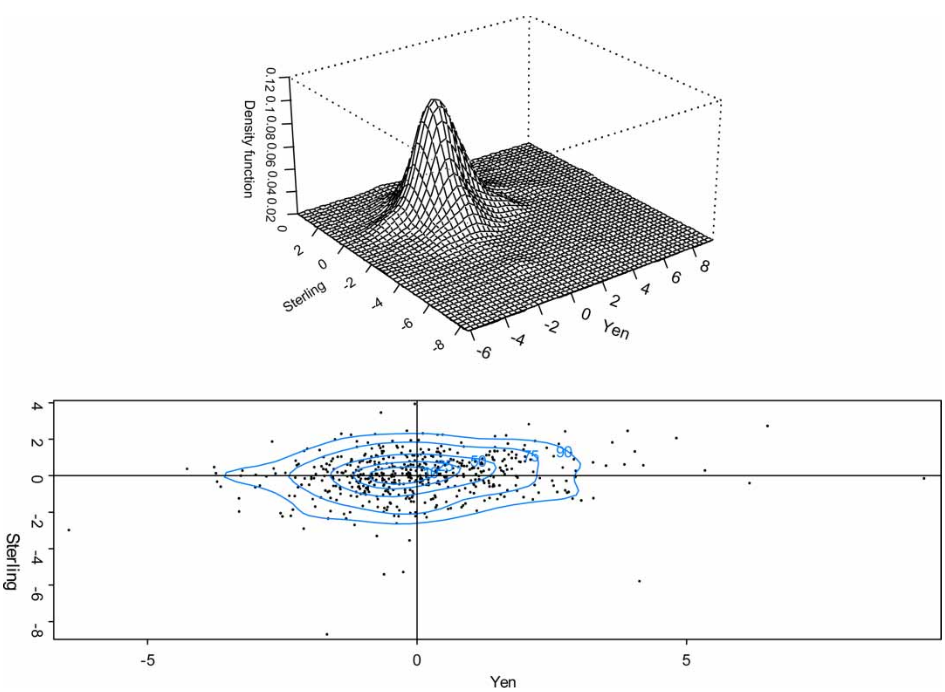

Figure 6. Empirical yen/sterling bivariate return distribution.

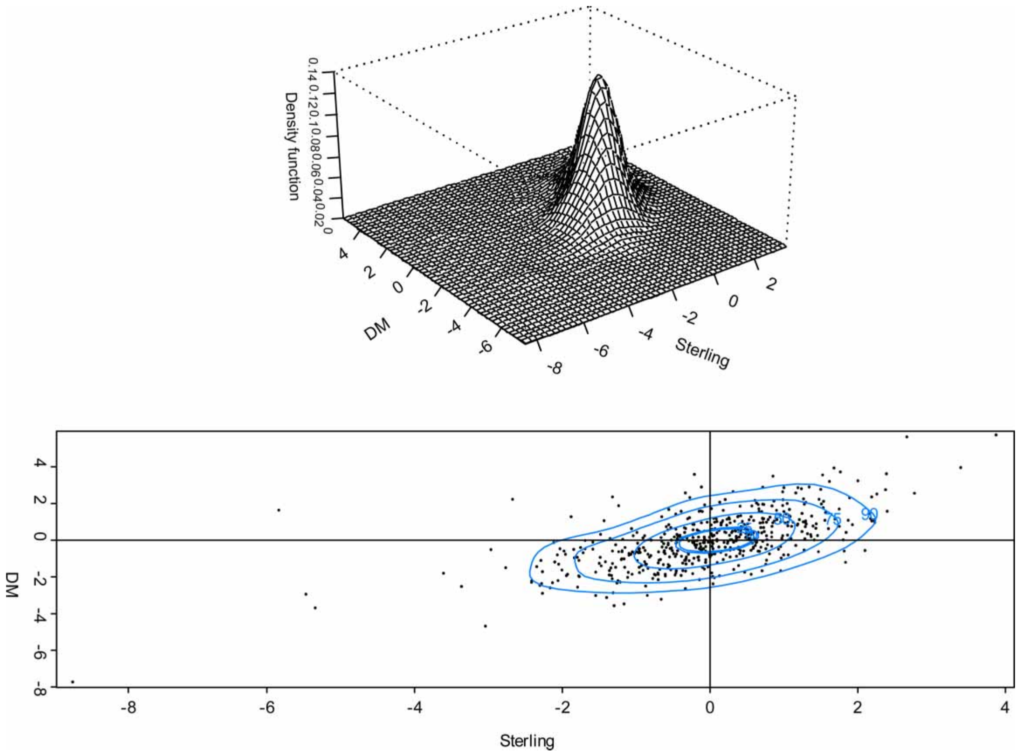

Figure 7. Empirical sterling/DM bivariate return distribution.

Table 5. Upper and lower tail index nonparametric estimates.

	λ_U^2		
λ_L^2	Yen	Sterling	DM
Yen	–	0.16	0.03
Sterling	0.09	–	0.32
DM	0.20	0.38	–

the parametric estimates in Table 6 using the relevant formulae for the Joe-Clayton copula (37), with the upper tail dependency parameters given above the main diagonal and the lower tail dependency given below.

These two sets of estimates differ in interesting ways; we can clearly see the moderate degree of both higher and lower tail dependence in both the nonparametric and parametric estimates for the DM:dollar and sterling:dollar rates but critically this is not strongly shown at the median parameter estimates. In fact, the upper tail dependency is shown only at the 5% quantile and not at all at the 95% quantile. Conversly, the lower tail dependence suggested by the nonparameteric estimate is only shown at the 95% quantile parameter estimates. The weak relationship between the yen and sterling dollar rates is shown effectively at all quantiles. The degree of both lower and upper nonparametric tail dependence between the yen and the DM rates is not found at

Table 6. Upper and lower tail dependencies using Joe–Clayton c-quantile regression parameter estimates: returns.

	r_t					
	USD/Y on USD/£		USD/Y on USD/DM		USD/£ on USD/DM	
p	λ_U	λ_L	λ_U	λ_L	λ_U	λ_L
5%	0.09	0.00	0.19*	0.00	0.37*	0.03*
10%	0.09	0.00	0.19*	0.00	0.35*	0.04*
50%	0.08	0.00	0.15	0.00	0.23*	0.15*
90%	0.06	0.00	0.12	0.04	0.09	0.27*
95%	0.05	0.00	0.11	0.05	0.08	0.28*

any quantile. It is, however, clear that we get substantially more information regarding the joint risk structure from carrying out this analysis using the c-quantile parameter estimates through being able to examine the dependence at all quantiles rather than simply through the mean. The question that is implicitly raised is whether we are really interested in asymptotic dependence or the dependence as shown by the quantile results at the particular level with which the risk manager may be concerned. Coles, Heffernan, and Tawn (1999) have also suggested that λ_U (and hence also λ_L) can be viewed as quantile-based by varying the level α in Equations (32) and (34) through the range [0, 1] as opposed to the normal limiting values at 0 and 1. It is not, however, enitrely clear if the interpretation of λ_U at a particular α corresponds to a quantile-based measure of *upper tail* dependence instead of simply a measure of quantile dependence. Carrying out their suggestion produces the results shown in Figures 8–13, where their χ and $\bar{\chi}$ statistics and 95% confidence intervals which correspond to our λ_U and $\bar{\lambda}$ statistics evaluated at each α value are shown.[6] The yen:sterling rates can be seen from these figures to be effectively independent except as we get close to the upper tail which contradicts our c-quantile results shown above. The yen:DM rates also appear to show weak dependence from these figures with somewhat more upper tail dependence as suggested by the quantile regression results above. The sterling:DM results show dependence that appears to decline as we get close to the upper tail and then explodes as we get to the tail; however, at this point, the confidence intervals are very wide. It would seem that the c-quantile approach is providing an alterantive and potentially more reliable view of tail area and moderate quantile dependence.

6.1 *Dynamic c-quantiles*

We next compute the nonlinear *dynamic* quantile regression estimates $(\hat{\delta}(p), \hat{\theta}(p))$ using the Joe–Clayton copula on returns so that we are examining the dependence between r_t and r_{t-1}:

$$\left(\hat{\delta}(p), \hat{\theta}(p)\right) = \arg\min \left(\sum_{t=1}^{T} \left(p - \mathbb{I}_{\{r_t \leq \mathbf{q}(r_{t-1}, p; \delta, \theta)\}}\right) (r_t - \mathbf{q}(r_{t-1}, p; \delta, \theta)) \right) \tag{42}$$

with

$$\mathbf{q}(r_{t-1}, p; \delta, \theta) = \hat{F}^{[-1]} \left[\phi_{\delta,\theta}^{-1} \left[\phi_{\delta,\theta} \left(\phi_{\delta,\theta'^{-1}} \left(\frac{1}{p} \phi_{\delta,\theta'}(\hat{F}(r_{t-1})) \right) \right) - \phi_{\delta,\theta}(\hat{F}(r_{t-1})) \right] \right] \tag{43}$$

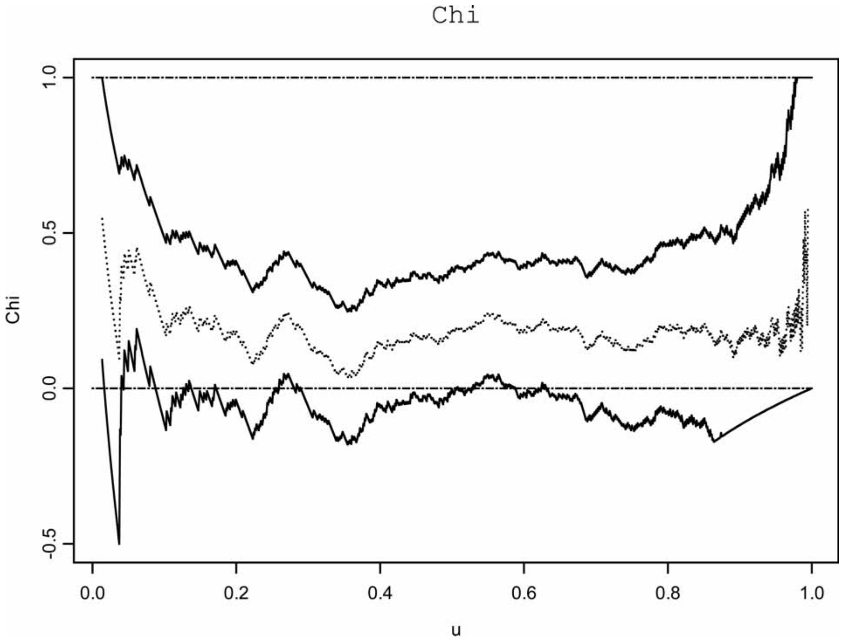

Figure 8. Yen:sterling estimates of λ for varying α.

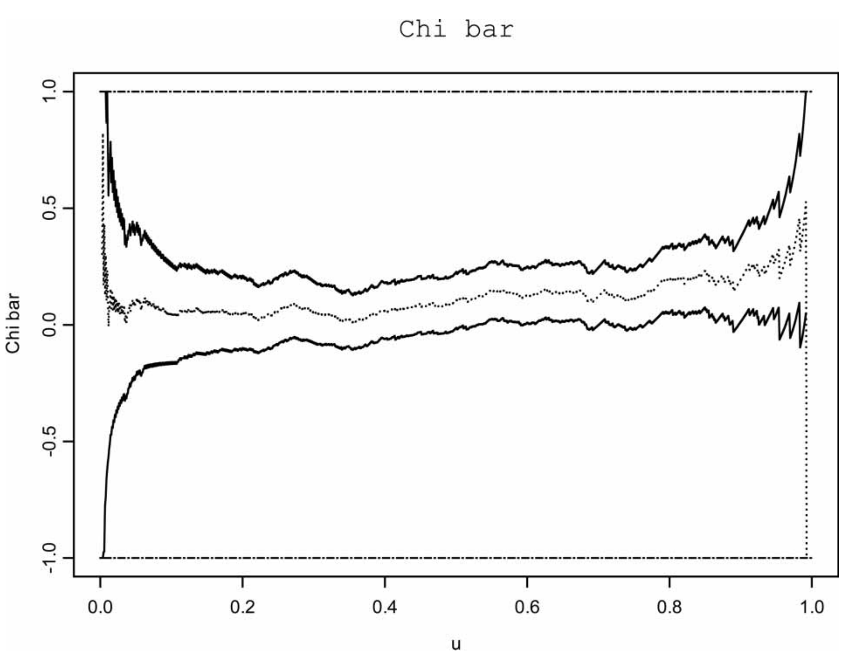

Figure 9. Yen:sterling estimates of $\bar{\lambda}$.

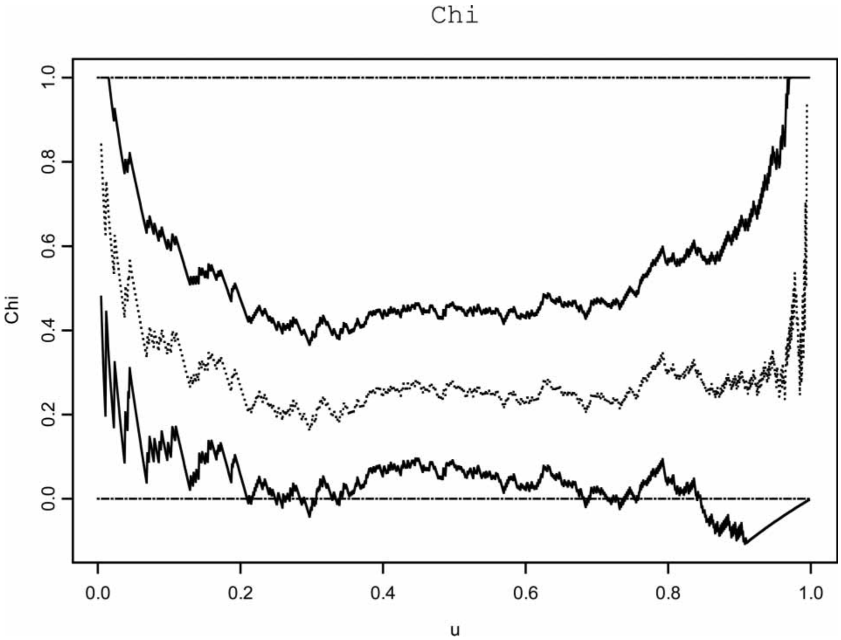

Figure 10. Yen:DM estimates of λ.

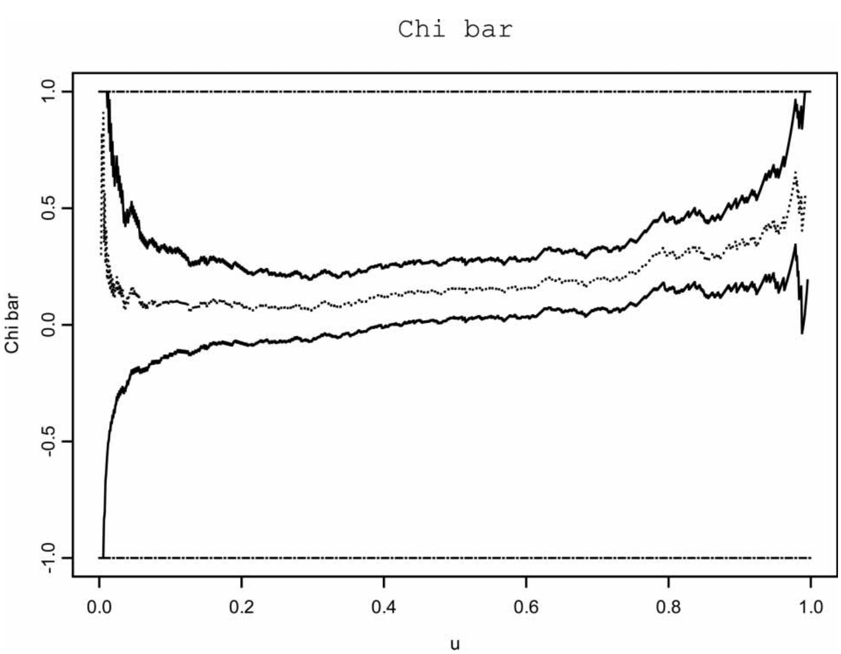

Figure 11. Yen:DM estimates of $\bar{\lambda}$.

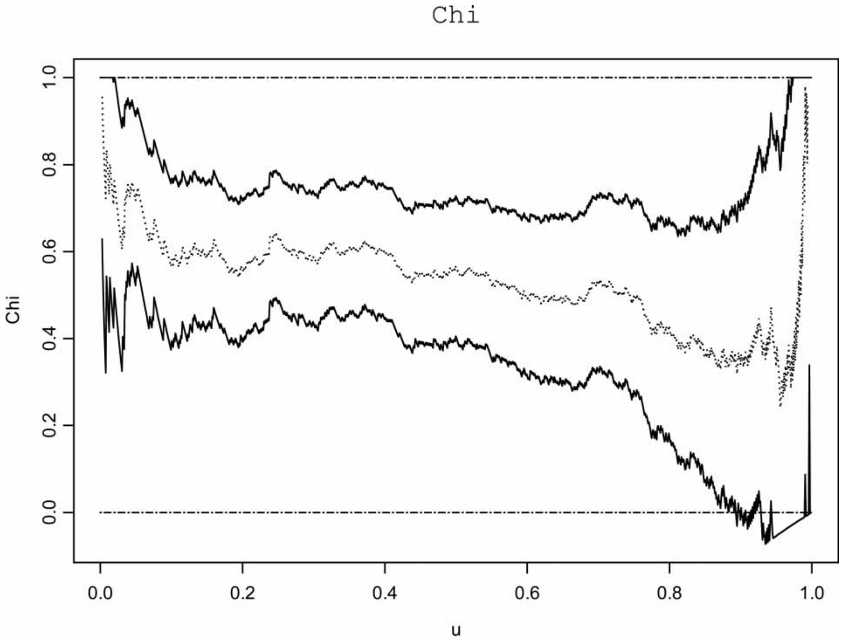

Figure 12. Sterling:DM estimates of λ.

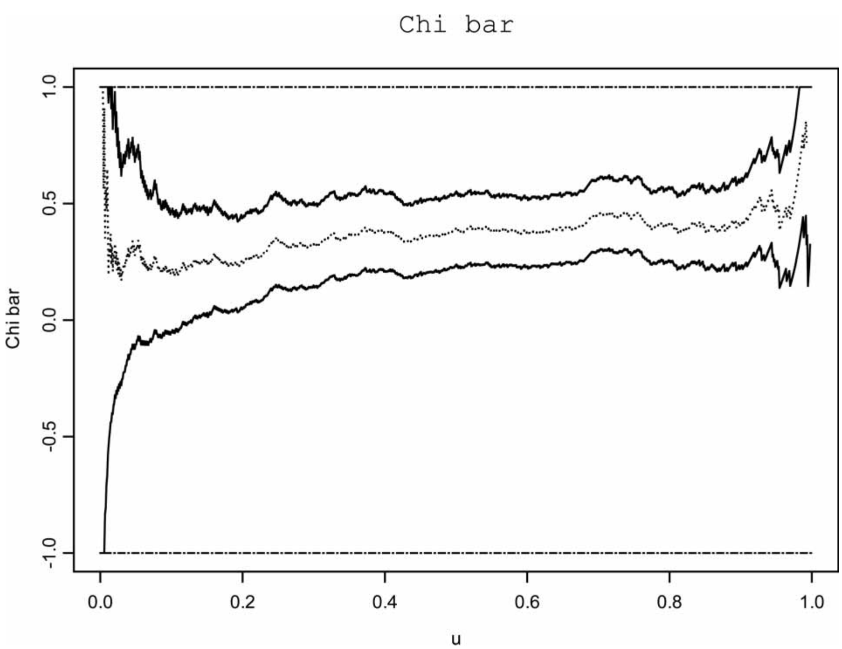

Figure 13. Sterling:DM estimates of $\bar{\lambda}$.

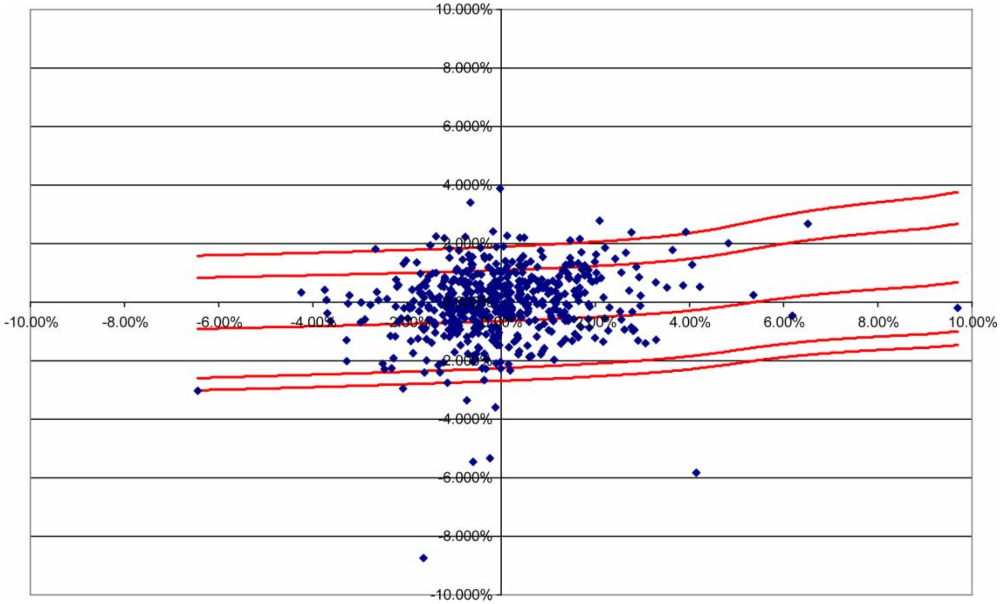

Figure 14. Non-parametric quantile regression of USD/Y on USD/£ for 5%, 10%, 50%, 90%, 95% probability levels.

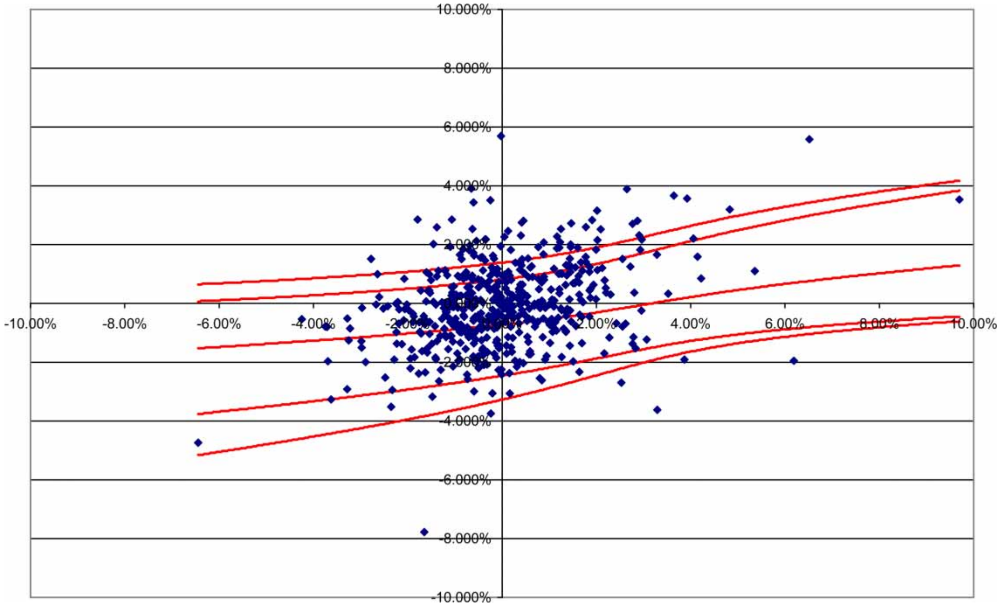

Figure 15. Non-parametric quantile regression of USD/Y on USD/DM for 5%, 10%, 50%, 90%, 95% probability levels.

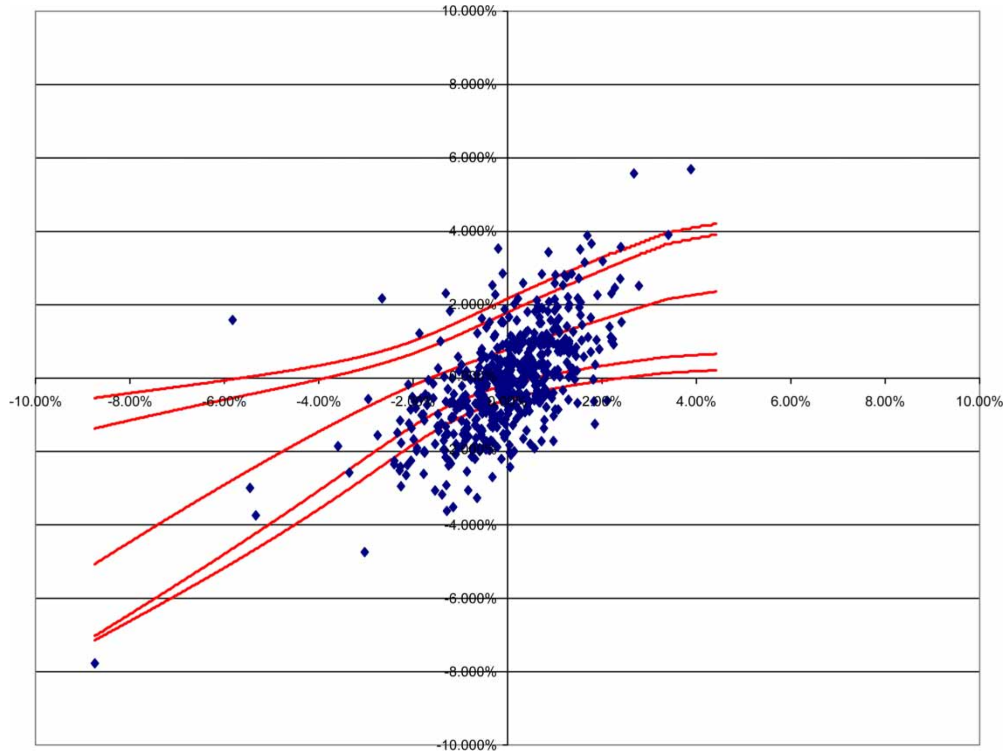

Figure 16. Non-parametric quantile regression of USD/£ on USD/DM for 5%, 10%, 50%, 90%, 95% probability levels.

Table 7. c-Quantile regression estimates of the relative return of the exchange rate $r_t = S_t/S_{t-1} - 1$ on r_{t-1}.

	r_{t-1}:r_t					
	USD/Y:USD/Y		USD/£:USD/Y		USD/DM:USD/Y	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.00	0.00	1.00	0.00	1.00	0.00
10%	1.01	0.00	1.00	0.00	1.00	0.00
50%	1.03*	0.00	1.00	0.00	1.00	0.00
90%	1.05*	0.00	1.01	0.00	1.01	0.00
95%	1.05*	0.00	1.01	0.00	1.02	0.00

with $\phi_{\delta,\theta}$ the generator of the copula defined in Equation (24) and $\hat{F}$ the empirical distribution function of the exchange rate return r_t. The estimates are given in Tables 7–9 below.

These results show that there is no significant dynamic dependence, either between cross-rates or within rates, at *any* quantile level between the returns of the exchange rates in this weekly data. The Joe–Clayton parameter estimates indicate independence even in the relative extremes of the joint distribution. This result appears to suggest that Forex markets retain efficiency, in a very standard sense, even when the markets are in crisis and in either the upper or lower tail. These

Table 8. c-Quantile regression estimates of the relative return of the exchange rate $r_t = S_t/S_{t-1} - 1$ on r_{t-1}.

	r_{t-1}:r_t					
	USD/Y:USD/£		USD/£:USD/£		USD/DM:USD/£	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.02	0.00	1.00	0.08	1.00	0.06
10%	1.02	0.00	1.00	0.07	1.00	0.06
50%	1.02	0.00	1.00	0.01	1.00	0.04
90%	1.02	0.00	1.00	0.00	1.02	0.00
95%	1.02	0.00	1.00	0.00	1.03*	0.00

Table 9. c-Quantile regression estimates of the relative return of the exchange rate $r_t = S_t/S_{t-1} - 1$ on r_{t-1}.

	r_{t-1}:r_t					
	USD/Y:USD/DM		USD/£:USD/DM		USD/DM:USD/DM	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.00	0.02	1.00	0.08	1.00	0.06
10%	1.00	0.02	1.00	0.07	1.00	0.06
50%	1.00	0.02	1.00	0.04	1.00	0.04
90%	1.00	0.01	1.00	0.00	1.02	0.00
95%	1.00	0.01	1.00	0.00	1.02	0.00

Table 10. Tail area dependency Measures on lagged own returns.

Tail area dependency	λ_L	λ_U	$\bar{\lambda}_U$	$\bar{\lambda}_L$
Yen	0.017	0.186	0.061	−0.016
DM	0.0	0.011	−0.054	−0.114
Sterling	0.097	0.0	−0.0613	0.126

quantile results are confirmed, but not quite so clearly, as shown in Table 10, when we examine the asymptotic tail area dependency measures;

7. Conclusion

In this paper, we have developed and applied a new approach to measuring dependence through copula quantile regressions. The methodology rests on identifying the copula that captures the dependence structure between the series of interest and then deriving the implied conditional quantile regression specification. This enables us to examine the conditional dependence of one variable conditional on the other(s) at a range of quantile levels as opposed to the normal regression relationship, which only describes the form of dependence at the conditional expectation. In this way, we can explore causal dependencies at moderate risk levels that may be more relevant to risk managers than the normal approach given by examining standard asymptotic tail area dependency measures. We have developed several theoretical results describing the properties of c-quantiles and compared their performance with the standard tail area dependency measures.

Our empirical results using exchange rates are indicative of the structure that can be uncovered using copula-based quantile regressions. We found similar patterns of symmetric and asymmetric dependence as reported in Patton (2006) using different techniques. We also found when examining dynamic copula-quantile dependence that the independence shown between the returns of one exchange rate on its own lag applies at all quantiles and hence a much stronger 'efficiency' condition seems to apply, even into the tails of the distribution than implied by standard martingale efficiency conditions that involve the conditional expectation. We have also noted the computational difficulties that are likely to complicate the standard measures of (asymptotic) tail dependence.

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Notes

1. Koenker and Bassett discuss properties of their estimator, especially through the following theorem:

THEOREM 1 *Let* $\beta^\star(p, y, X) \in \mathcal{B}^\star(p, y, X)$. *Then, the following properties hold*:

(1) $\beta^\star(p, \kappa y, X) = \begin{cases} \kappa\beta^\star(p, y, X) & \text{for } \kappa \in \mathbb{R}^+ \\ \kappa\beta^\star(1-p, y, X) & \text{for } \kappa \in \mathbb{R}^- \end{cases}$,

(2) $\beta^\star(p, y + X\delta, X) = \beta^\star(p, y, X)$ *for* $\delta \in \mathbb{R}^k$,

(3) $\beta^\star(p, y, X\Gamma) = \Gamma^{-1}\beta^\star(p, y, X)$ *with* Γ *non-singular* $(k \times k)$ *matrix.*

2. Note that C_1 has to be partially invertible in its second argument. If it is not analytically invertible, a numerical root-finding procedure can be used.
3. Again for simlicity, we start by examining the dependence between the level of the exchange rates while recognizing that on the basis of some statistical criteria exchange rate levels may appear to be nonstationary. However, it is in principle clear that exchange rates cannot follow a stochastic process with an infinite variance which can also take negative values.
4. It is natural that we find contemporaneous dependence between the exchange rates given the presence of triangular arbitrage relationships in the market.
5. The Joe–Clayton copula was preferred by the data in AIC comparisons with several alternative copulae including the Gaussian copula.
6. We are grateful to Jan Heffernan for making the SPLUS code for computing these figures publically available.

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