Doubly Stochastic Binomial Point Processes and Time Deformation in Financial Markets

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Search for a Model that Explains the Stylized Facts of Financial Market Returns.

(i) Leptokurtosis.

(ii) Long range dependence in absolute values of returns and squares of returns but not the levels.

(iii) Volatility clustering.

(iv) Aggregational Gaussianity.

(v) Leverage effect.

The Background

- 1960's Fama(1963), Mandlebrot(1963) clearly rejected Gaussianity in log returns Mandlebrot also noted persistence of volatility, volatility clustering and suggested that apparent scaling and self similarity results indicated the use of stable families in place of Gaussian. Mandlebrot and Taylor (1967) Subordination model with a *stable* law with index $\alpha/2$ (where $\alpha < 2)$. Infinite second moment and no LRD. infinite downside risk and does not match the data.
- 1970's Clark (1973) Stochastic Volatility, Time deformation and rejection of infinite variance and Stable family model– log normal- related transaction time to trading volume- couldn't get the moments right and no long memory.
- 1980's Development of Stochastic Volatility, Taylor (1986) and Engle (1982) conditional heteroskedasticity models-

Returns: Long Memory, Time Deformation

- 1990's- Volume -Volatility relationship and information flow; a number of different economic measures of market time were considered; le Fol - volume, Ané and Geman (2000) - number of trades.
- Advent of transaction level and high frequency data, location- scale mixture models for Stochastic Volatility and time deformation- eg.

  (i) *Variance Gamma*

  (ii) *Normal Inverse Gaussian*

  (iii) *Generalised Hyperbolic*

  - See Barndorff-Nielsen(1977), Mencia and Sentana(2005)- $X = \alpha + \beta\varsigma^{-1} + \varsigma^{-1/2}\Upsilon^{1/2}u$ where u is a spherical random normal vector and ς is a Generalised Inverse Gaussian $\sim GIG(-\nu, \gamma, \delta)$ encompasses the Variance-Gamma, NIG and Hyperbolic processes

  (iv) *CGMY*,

  (v) *Meixner* (see Wim Schoutens, Lévy Processes in Finance).

Returns: Long Memory, Time Deformation

- 2000's Point process models, slightly different objective
  (i) *Duration*- ACD+... Engle and Russell
  (ii) *Intensity models* - ACI, Russell, Hawkes Processes-Bowsher, Hautsch
  (iii) *Count* models
  (iv) *OU models driven by Lévy Processes*, Barndorff Nielsen and Shephard, Poisson, Compound Poisson, Gamma, Inverse Gaussian, etc in place of Brownian Motion
  (v) Pure Jump processes CGMY, Jump diffusions...Where N_t is say a Poisson process counting the jumps (Information events) and need the distribution of jumps Y_t.

$$X_t = \gamma t + \sigma W_t + \sum_{t=1}^{N_t} Y_t$$

Returns: Long Memory, Time Deformation

- Research now expanding on 3 fronts- *Dynamic Point process* models, *extensions of Lévy process models* and *Compensators* as opposed to Subordinators and role of jumps.
  - **Our objective is to build the time deformation process up from the data and in doing so we link the point process and time deformation literatures.**
- In particular evidence appears to be accumulating that the time deformation process is time dependent and has long memory- it is clear that Lévy processes which imply independent increments *cannot* explain asset prices.
- This implies the integrated intensity process, $\Lambda(t)$, and any other measures of market activity should show Long Range Dependence. See Mandlebrot, Fisher Calvet (1997a,b), Marinelli, Rachev and Roll (2003 a,b) Stable Family LRD, Heyde (1999)....... our starting point
- We are investigating the use of a stochastic time change without the independent increments assumption- a compensator with LR dependence– specifically we will model this as a Hawkes process with Hyperbolic memory.

Returns: Long Memory, Time Deformation

- The initial paradigm: *Geometric Brownian Motion* model the price, P_t, at time t as $P_t = P_0 \exp[\mu t + \sigma W(t)]$, where $\mu, \sigma > 0$ are fixed constants and $W(t)$ is standard Brownian motion.
- Then log returns are i.i.d. Gaussian with mean μ and variance σ^2., $X_t = \log P_t - \log P_{t-1} = \mu + \sigma(W(t) - W(t-1))$.
- Following Heyde(1999)- Instead we consider a random time changed version of Geometric Brownian Motion$P_t = P_0 \exp[\mu t + \sigma W(T_t)]$, where $\{T_t\}$ is a positive increasing random process with stationary differences which is assumed to be *independent* of the Brownian motion $\{W(t)\}$ and the differences of the $\{T_t\}$ are Long Range Dependent and have heavy tails. Similar assumptions have been made by C.Heyde (1999), Mandlebrot et al. (1997) and Marinelli et al (2003).
- Consider $T_t \sim t$ almost surely as $t \to \infty$ then

$$\begin{aligned} X_t &= \log P_t - \log P_{t-1} = \mu + \sigma(W(t) - W(t-1)) \\ &\stackrel{d}{=} \mu + \sigma(T_t - T_{t-1})^{1/2} W(1) \end{aligned}$$

the heavy tails come from $\tau_t = T_t - T_{t-1}$.

Research

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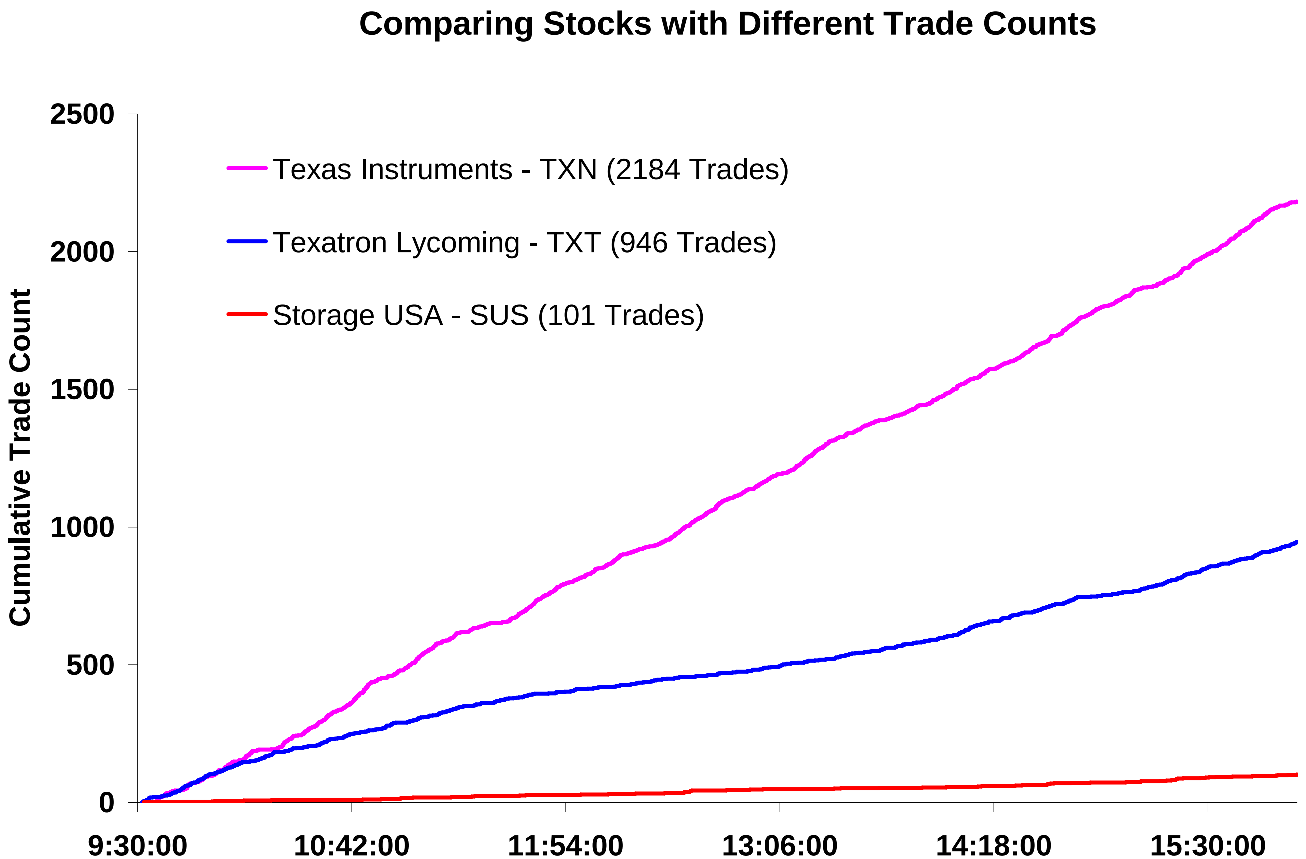
- In particular, use the DSBPP to show Long-Memory (Hurst = 0.7) in NYSE ‘Market Time’.
- Review the role of Mathematica in the Research.

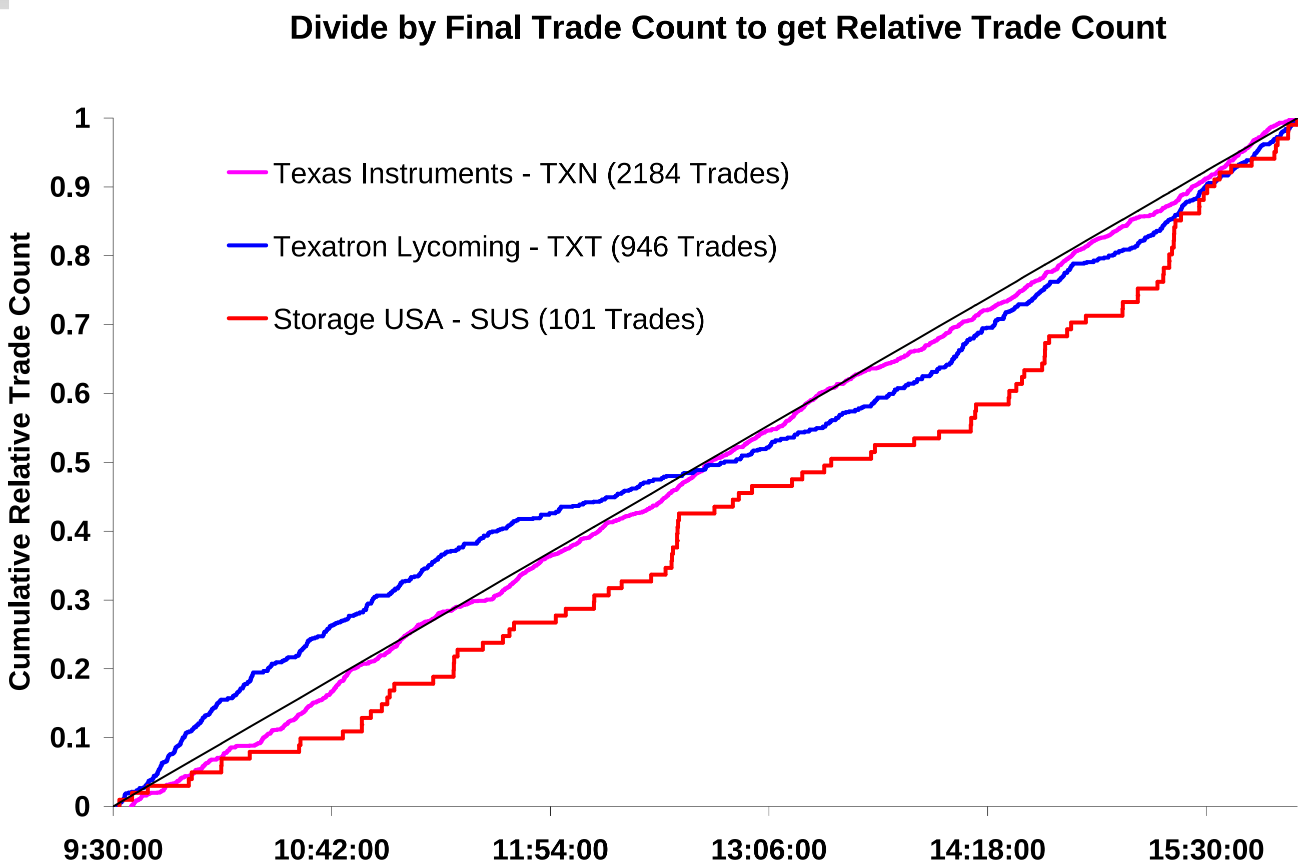
Empirical Relative Trade Counts

Ané and Geman (2000) - The ‘Market Clock’ is Trade Count Distribution.



Empirical Relative Trade Counts

Relative Count Measure $R(t) = \frac{N(t)}{N(T)},\ t \in [0, T]$



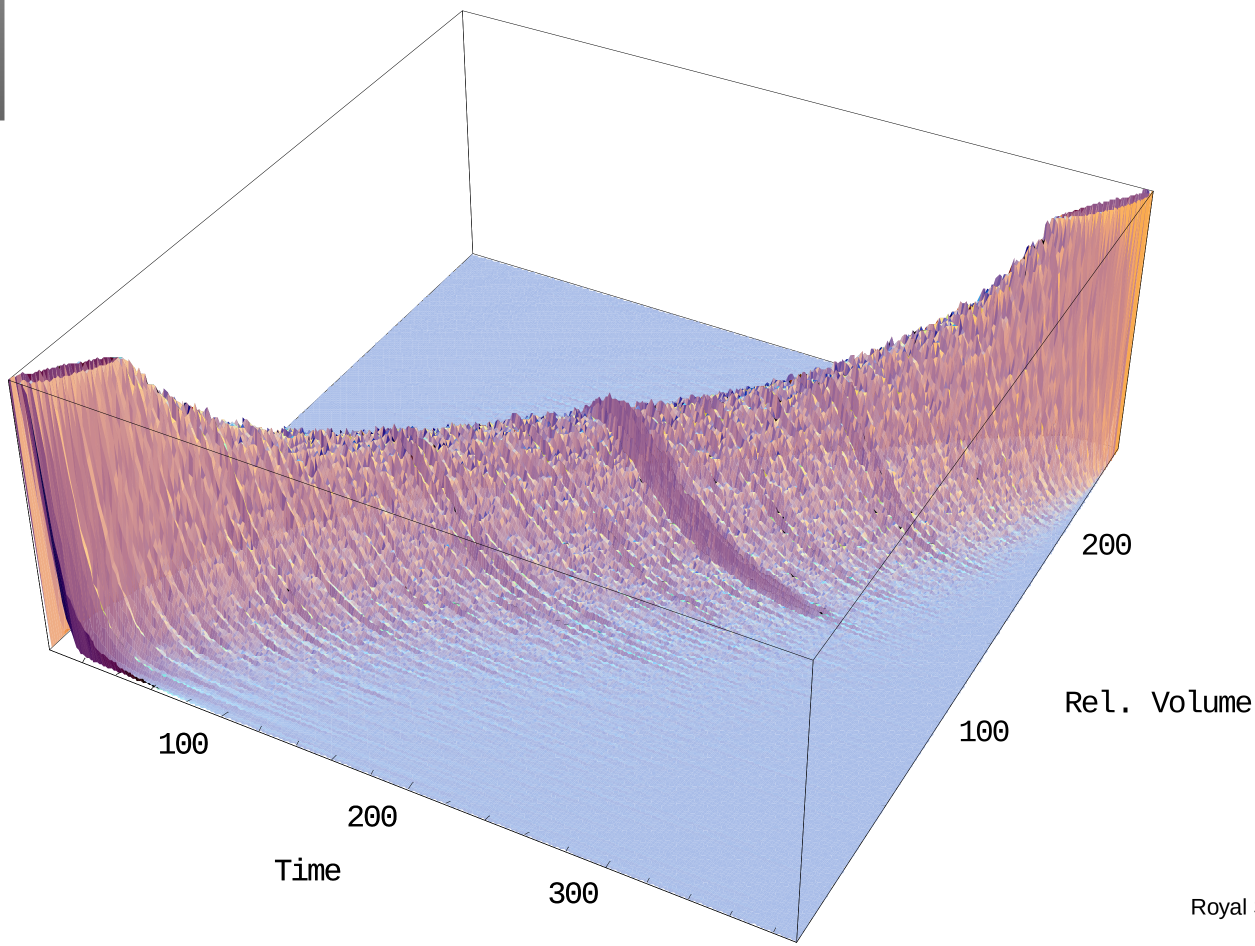
Empirical Relative Trade Counts

Relative Count Measure $R(t) = \frac{N(t)}{N(T)}$

- $N(T) = K \implies R(t) = a \in \{0, \frac{1}{K}, \ldots, \frac{K-1}{K}, 1\}$
- $R(0) = 0$
- $R(T) = 1$
- $R(t) \in [0, 1], \quad t \in (0, T)$
- $t_1 \leq t_2 \implies N(t_1) \leq N(t_2) \implies R(t_1) \leq R(t_2)$

Empirical Relative Trade Counts

90,000 NYSE Relative Volume Trajectories
$R(t),\ t \in [0, T]$, 391×253 Histogram



Empirical Relative Trade Counts

Mathematica Visualization- Relative Count Trajectories, $R(t)\ t \in [0, T]$.

Stocks ≥ 50 Trades per Day.

- Rotate $R(t)$ Trajectories about the Z-axis.
- Rotate $R(t)$ Trajectories about the Y-axis.
- Re-scale the Z-axis of the $R(t)$ Trajectories.
- Time Slice the $R(t)$ Trajectories.

Stocks ≤ 50 Trades per Day.

- Time Slice the $R(t)$ Trajectories.

Empirical Relative Trade Counts

The Distribution of the Relative Count Measure
$R(t) \sim \Theta_t(k)$

- $\Theta_t(k)$ Discrete (Binomial Superposition)
- $N(T) = K$, Domain$\left[\Theta_t(k)\right] = k \in \{0, \frac{1}{K}, \ldots, \frac{K-1}{K}, 1\}$
- $t = 0, \quad \Theta_0(0) = 1$
- $t = T, \quad \Theta_T(1) = 1$
- $t_1 \leq t_2 \implies E\left[\Theta_{t_1}(k)\right] \leq E\left[\Theta_{t_2}(k)\right]$

The Binomial Point Process

A point process is based on the probability of arrival in an interval Δ, $Pr\big[N(t+\Delta) - N(t) > 0|\mathcal{F}_t\big]$.

- Counting Measure $N(t)$
- Intensity $\lambda(t) = \lim_{\Delta \to 0^+} \frac{Pr[N(t+\Delta)-N(t)>0|\mathcal{F}_t]}{\Delta} \geq 0$
- Integrated Intensity $\Lambda(t) = \int_0^t \lambda(s)\,ds$ (non-decreasing).
- $\Lambda(t) = E[N(t)]$, $\Lambda(t) - N(t) = M_t$

Trade Arrivals are a Cox Process

- The Intensity $\lambda(t) \geq 0$ and Integrated Intensity $\Lambda(t)$ are random processes.
- Trade Count Measure $N(t)$ Conditionally Poisson Distributed.

$$Pr\{N(t) = K \,|\, \Lambda(t)\} = \frac{\exp[-\Lambda(t)]\,\Lambda(t)^K}{K!}$$

The Relative Count Measure, $R(t)$, is a Doubly Stochastic Binomial Point Process.

$$Pr\{R(t) = a \,|\, \Lambda(T)\} \;=\; Pr\{N(t) = aK \,|\, N(T) = K, \Lambda(T)\}$$

$$= \binom{K}{aK} \left[\frac{\Lambda(t)}{\Lambda(T)}\right]^{aK} \left[1 - \frac{\Lambda(t)}{\Lambda(T)}\right]^{(1-a)K}$$

$$a \in \{0, \frac{1}{K}, \ldots, \frac{K-1}{K}, 1\}, \quad t \in [0, T]$$

The Components of the Doubly Stochastic Binomial Point Process.

- Binomial Noise (Uninteresting)

$$\binom{K}{aK}\left[\bullet\right]^{aK}\left[1-\bullet\right]^{(1-a)K}, \quad a \in \{0, \frac{1}{K}, \ldots, \frac{K-1}{K}, 1\}$$

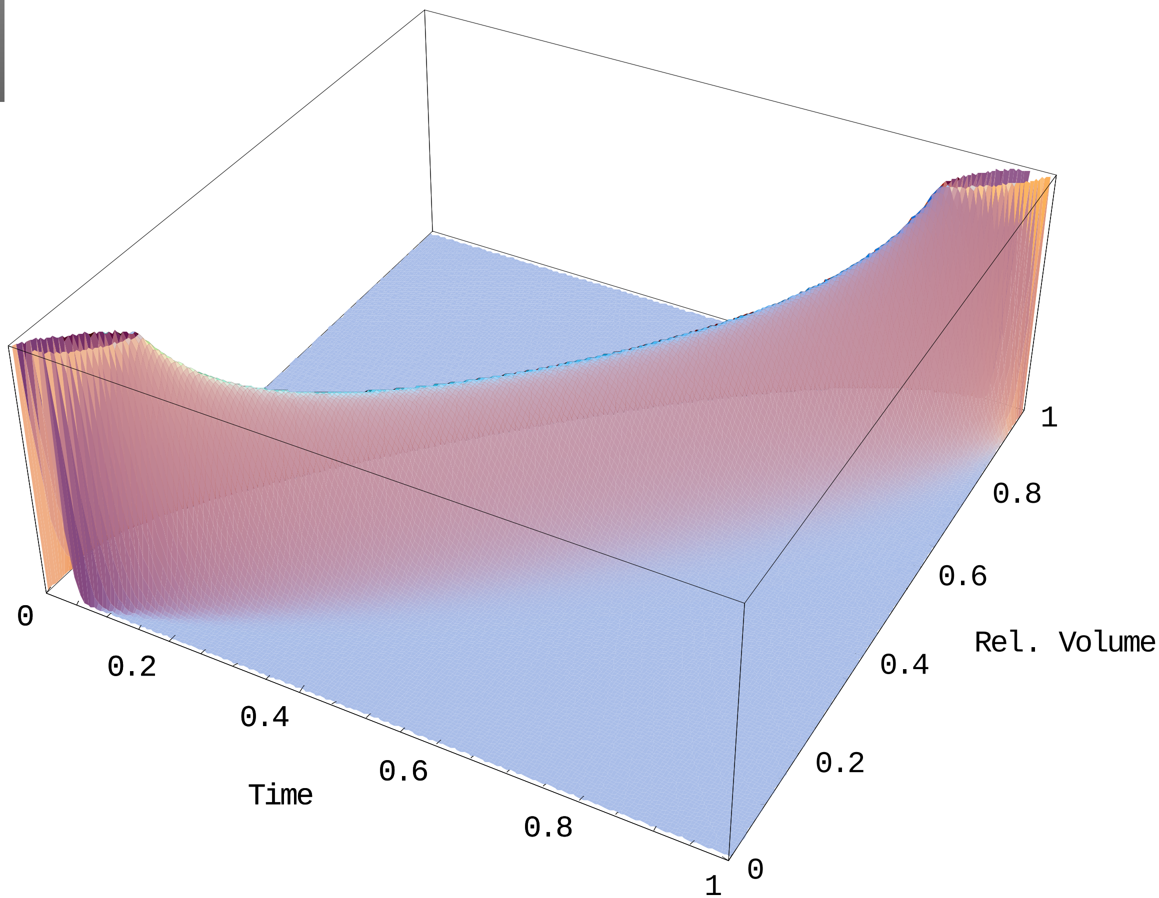
- The Self-Normalized Integrated Intensity - Random Probability Measure (Very Interesting)

$$\bullet \equiv \frac{\Lambda(t)}{\Lambda(T)}, \quad t \in [0, T]$$

The Random Probability Measure

$\frac{\Lambda(t)}{\Lambda(T)}$ is a smooth version of $R(t)$.

Simulated Using Mathematica.



The Random Probability Measure

Fubini's Thm on Product Prob. Spaces $\Longrightarrow$ $R(t) \sim$ Binomial Mixture

$$Pr\{R(t) = a\} \; = \int_0^1 Binomial(K, s)\, \Phi_t(s) ds$$

$\dfrac{\Lambda(t)}{\Lambda(T)}$ is a Time Indexed set of Distributions

- $t = 0 \;\; \Longrightarrow \;\; \frac{\Lambda(0)}{\Lambda(T)} \sim$ Dirac$\delta(z), \quad z \in [0, 1]$
- $t = T \;\; \Longrightarrow \;\; \frac{\Lambda(T)}{\Lambda(T)} \sim$ Dirac$\delta(z - 1), \quad z \in [0, 1]$
- $t \in (0, T) \;\; \Longrightarrow \;\; \frac{\Lambda(t)}{\Lambda(T)} \sim \Phi_t(z), \quad z \in [0, 1]$

RPM Moments

$Pr(R(t) = a) \sim \Theta_t(a)$ a Binomial Mixture. Solve for the raw moments of the RPM distribution $\Phi_t(s)$ in terms of the data distribution $\Theta_t(a)$ raw moments.

$$\sum_{i=0}^{K}\left(\frac{i}{K}\right)^n \Theta_t(i) = \sum_{i=0}^{K}\left(\frac{i}{K}\right)^n \int_0^1 \text{Binomial}(K, s)\, \Phi_t(s) ds$$

$$= \frac{1}{K^n}\int_0^1 \left[\sum_{i=0}^{K} i^n \binom{K}{i} s^i (1-s)^{K-i}\right] \Phi_t(s) ds$$

$$E[\Theta_t(a)] = \sum_{i=0}^{K}\left(\frac{i}{K}\right)\Theta_t(i) = \frac{1}{K}\int_0^1 (Ks)\, \Phi_t(s) ds = E[\Phi_t(s)]$$

Re-programming Mathematica

The 4th Raw Moment of $\Phi_t(s) \sim \frac{\Lambda(t)}{\Lambda(T)}$ was calculated by re-programming the Integrate function in Mathematica.

- δ_i is the ith Raw Moment of (observed binned data) $\Theta_t(k|K) \sim R(t)$
- λ_i is the ith Raw Moment for $\Phi_t(s) \sim \frac{\Lambda(t)}{\Lambda(T)}$
- Final Trade count K.

$$\lambda_4 = \frac{K^3\delta_4 - 6K^2\lambda_3 + 18K\lambda_3 - 12\lambda_3 - 7K\lambda_2 + 7\lambda_2 - \lambda_1}{K^3 - 6K^2 + 11K - 6}$$

```
Unprotect[Integrate]; (* Add Rules to Integrate*)

Integrate[(x_ + y_)*z_,s_] := Integrate[x*z,s] + Integrate[y*z,s];

Integrate[x_ + y_,s_] := Integrate[x,s] + Integrate[y,s];

Protect[Integrate]; (* Prevent Any Modification of Integrate *)
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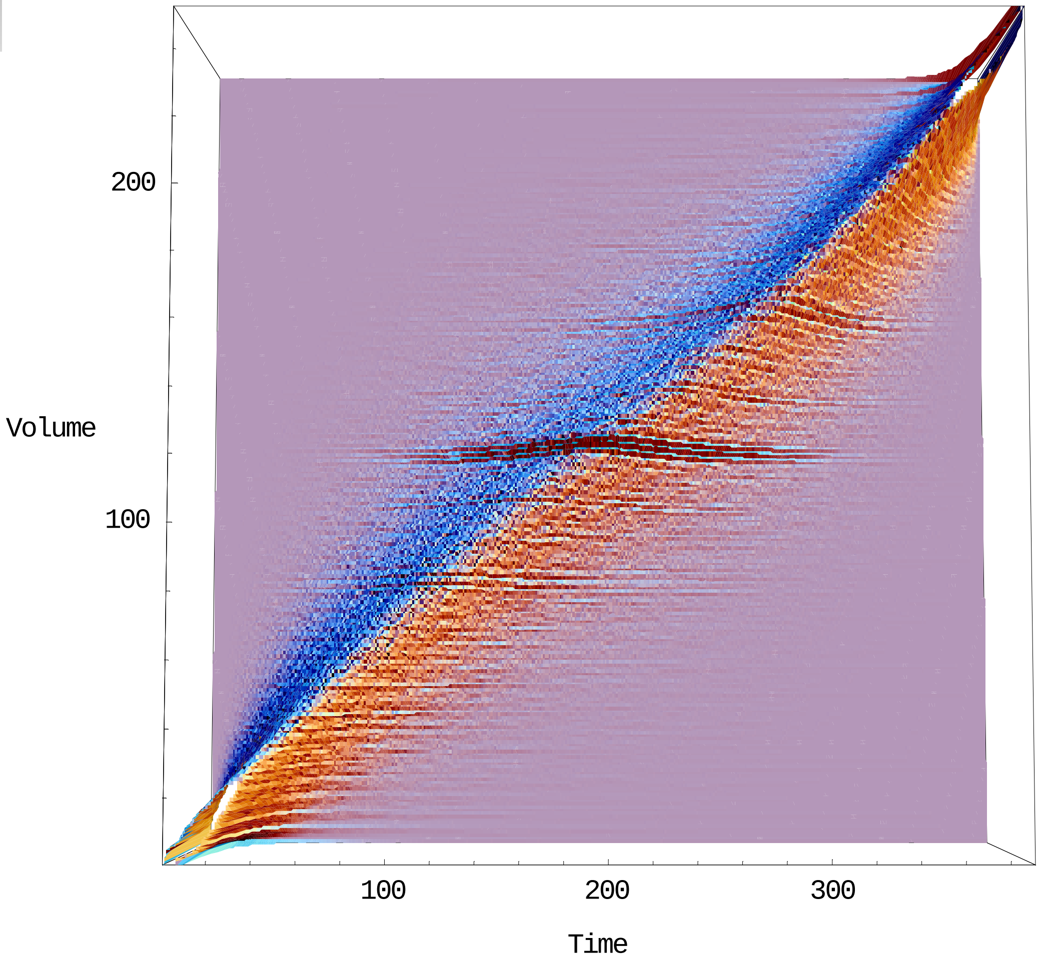
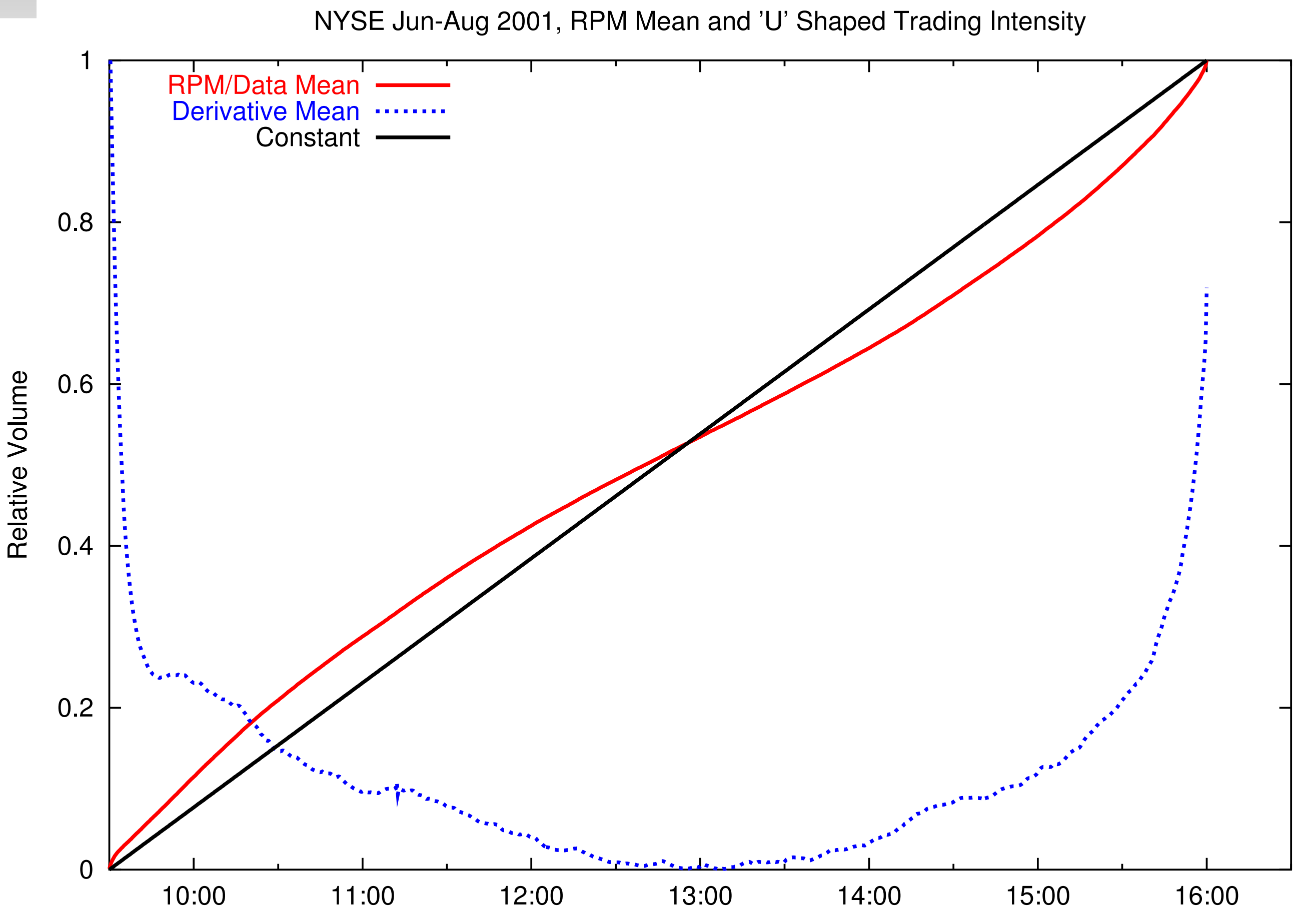
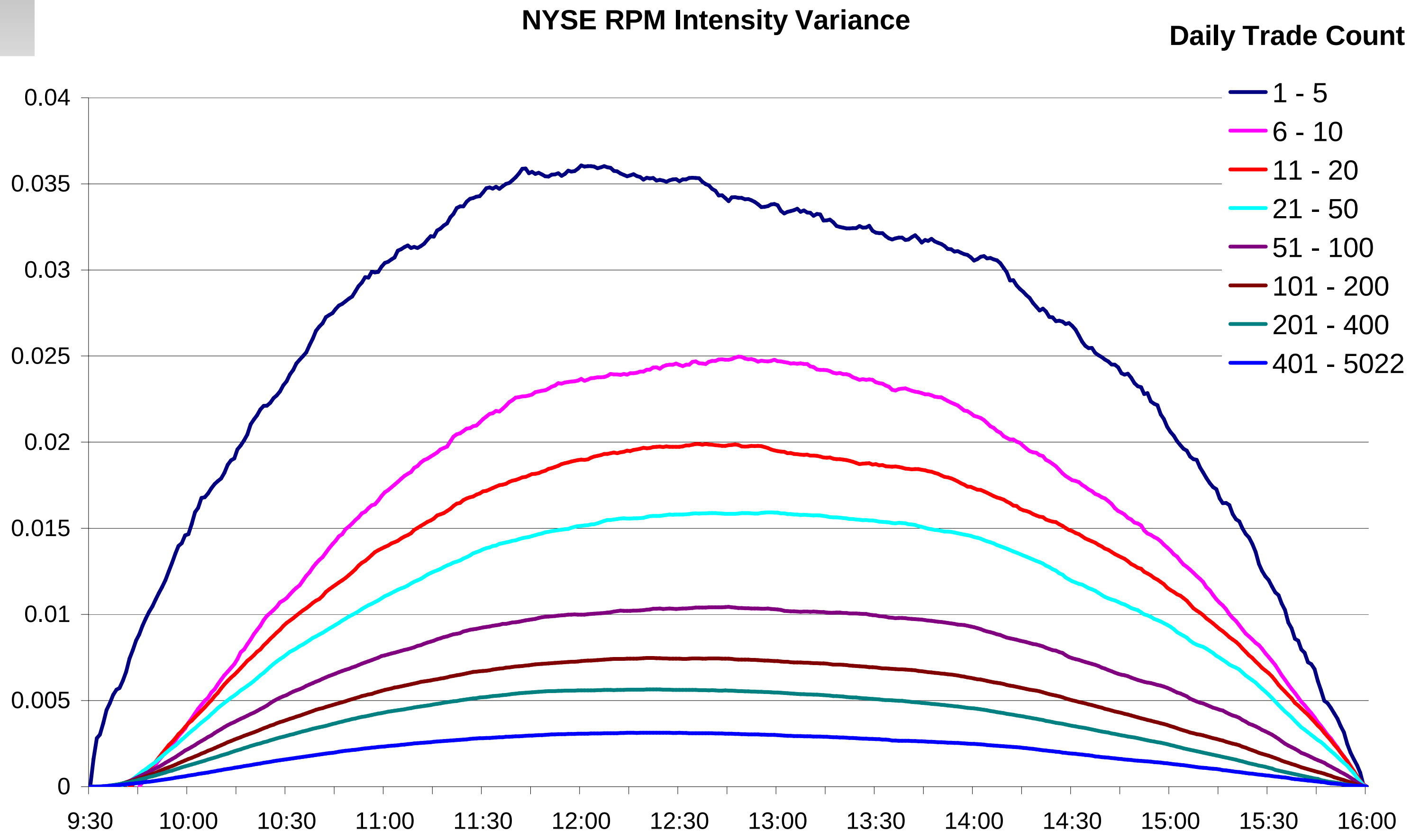
391(time) $\times$ 253(rel. vol.) Histogram (≥ 50 Trades)



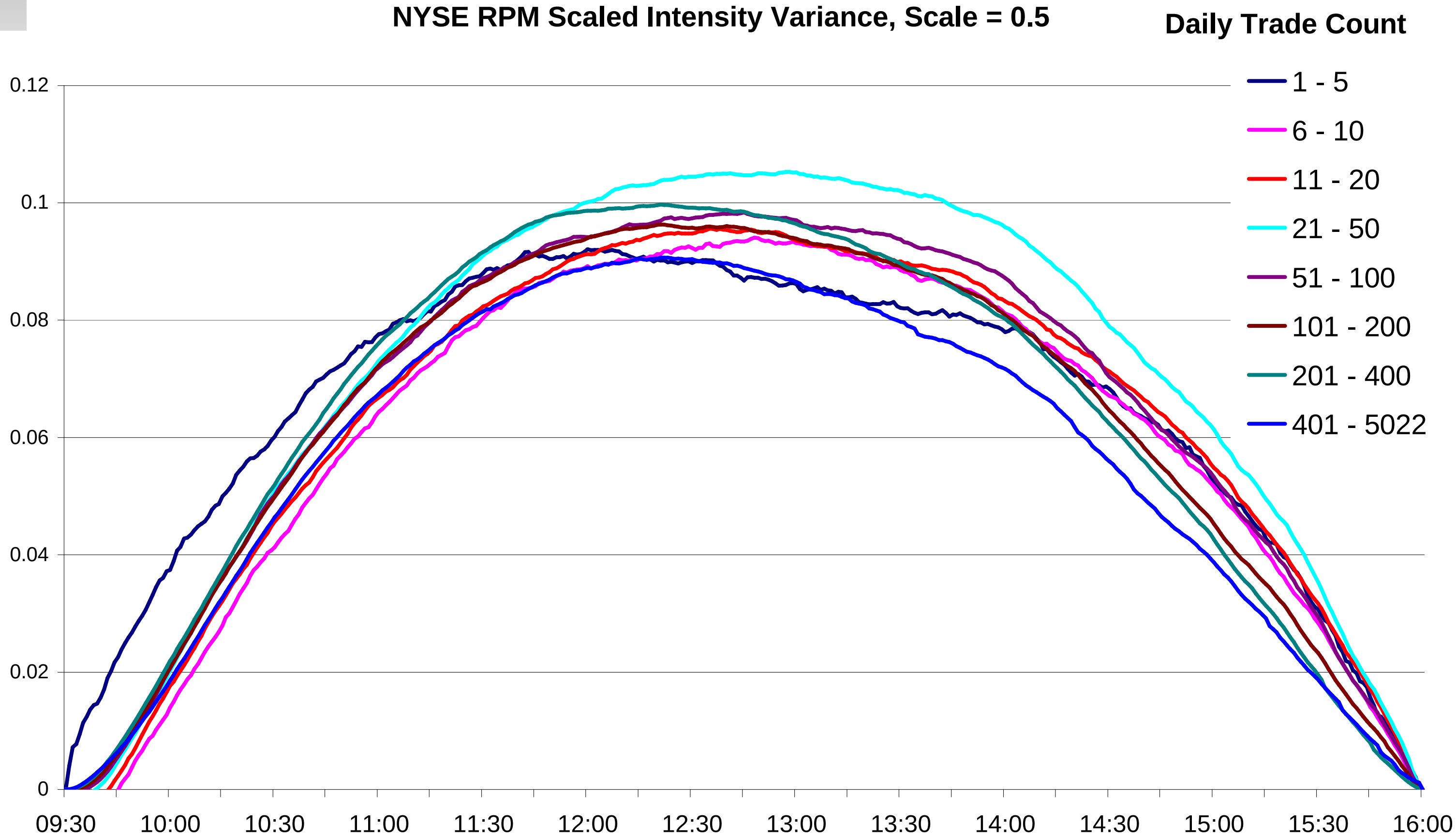
Derivative of $E[\frac{\Lambda(t)}{\Lambda(T)}]$ is the 'U' shaped trading variation.



RPM Data Variance for different Trade Count Bands

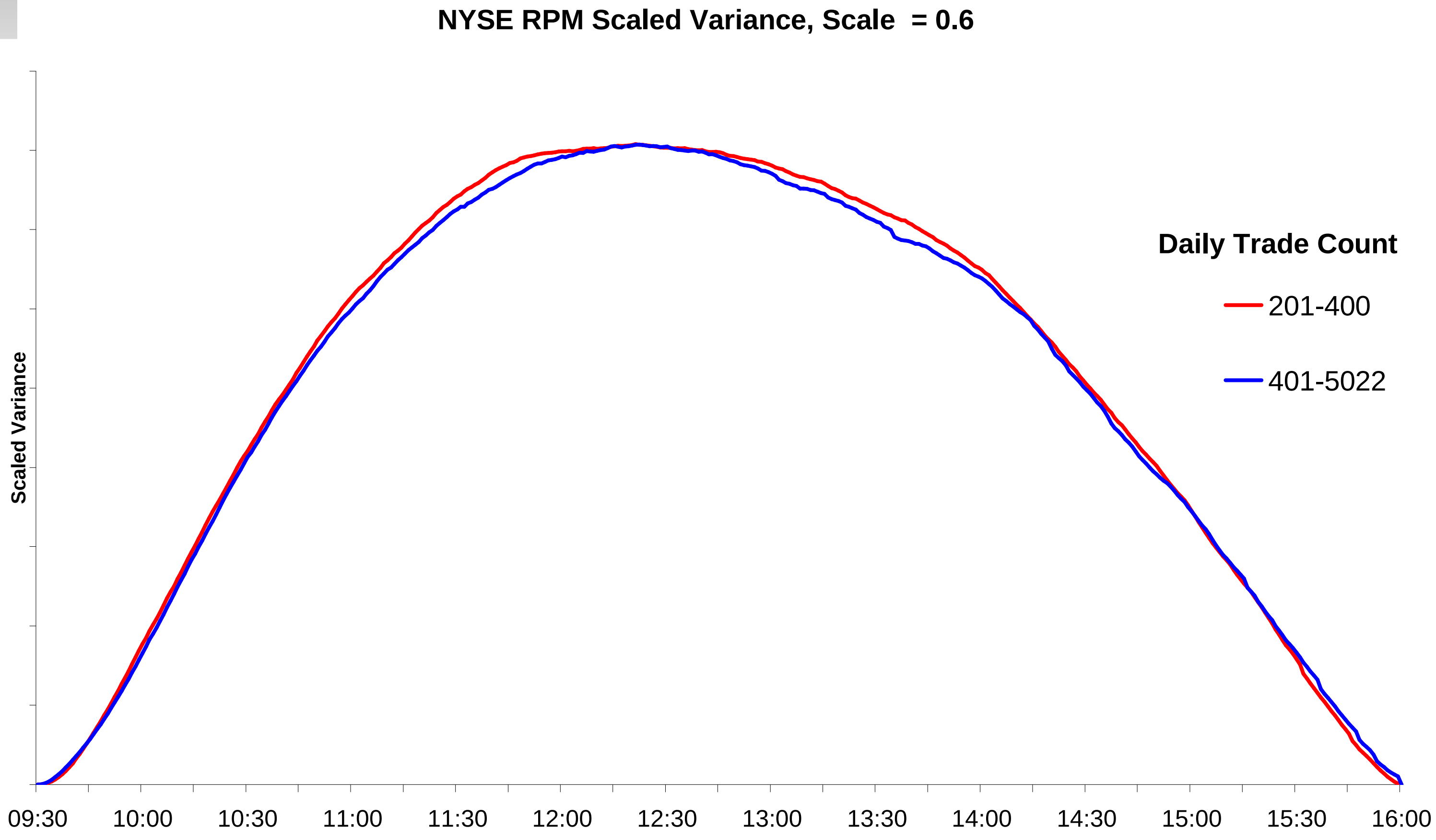


RPM Variance scaled by $(\sqrt{\lambda}+1)$



RPM Moments

Variance scaled by $(\lambda^{0.6}+1) \implies$ Self-Similar



Sketch Proof that the Intensity Measure $\Lambda(t)$ has a Hurst Exponent of 0.7

- Assume the Integrated Intensity is 2nd Order Self-Similar

$$\text{Var}\left[\Lambda(\lambda t)\right] \;=\; \lambda^{2H}\text{Var}\left[\Lambda(t)\right]$$

- The Taylor series approximation of a ratio of random variables gives:

$$\text{Var}\left[\frac{\Lambda(\lambda t)}{\Lambda(\lambda T)}\right] \approx \frac{1}{\lambda^{2H-2}+1}M, \quad M\text{constant in } \lambda$$

- Therefore $2H - 2 = -0.6$ and $H = 0.7$.

Summary

- Relative Trade Count is a Binomial Point Process directed by a Self-Normalized Integrated Intensity. We believe this is a novel object in Statistical Research.
- Modelling the Self-Normalized Integrated Intensity (RPM), $\frac{\Lambda(t)}{\Lambda(T)}$, gives important insights into the time structure of the Integrated Intensity, $\Lambda(t)$, and Intensity Process, $\lambda(t)$ that is Completely Free of Modelling Assumptions.
- One Insight is that The Hurst Exponent of the Integrated Intensity Aggregated Across Stocks, $\Lambda(t)$, is 0.7
- This invalidates all models of Market Returns that use Stochastic ‘Market Clocks’ (Subordinators) with $H = 0.5$ including the *Variance Gamma, Normal Inverse Gaussian and Generalized Hyperbolic* return distribution models.
- $\Lambda(t)$ Second order self-similar $\implies$ $B[\Lambda(t)]$ fits stylized facts (Clark 1973,Mandelbrot, Fisher & Calvet 1997).

Mathematica Summary

- Visualization of Complex Objects.
- Flexibility - The Ability to Re-programme Underlying Functionality.
- MathStatica - Simplifies Difficult and Time Consuming Mathematical Statistics.
- Easy Simulation of Complex Processes.