Financial Applications of Copulae

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Financial Econometrics Research Centre

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Practical Finance is faced with two basic problems

- Returns are non-gaussian
- Need to consider joint risks or *multivariate* distributions
  - often making standard mean variance results very poor approximations to optimal results-
- This applies virtually everywhere ...Risk Management, Pricing, Hedging, Portfolio construction,.....

The Copula

- A copula is simply a function that links univariate marginals to their joint multivariate distribution or alternatively it is a joint distribution function with uniform marginals.

$$C(u_1, u_2, \ldots, u_N) = \Pr[U_1 \leq u_1, U_2 \leq u_2, \ldots, U_N \leq u_N]$$

  with $U_1, U_2, \ldots, U_N$ being uniform random variables.

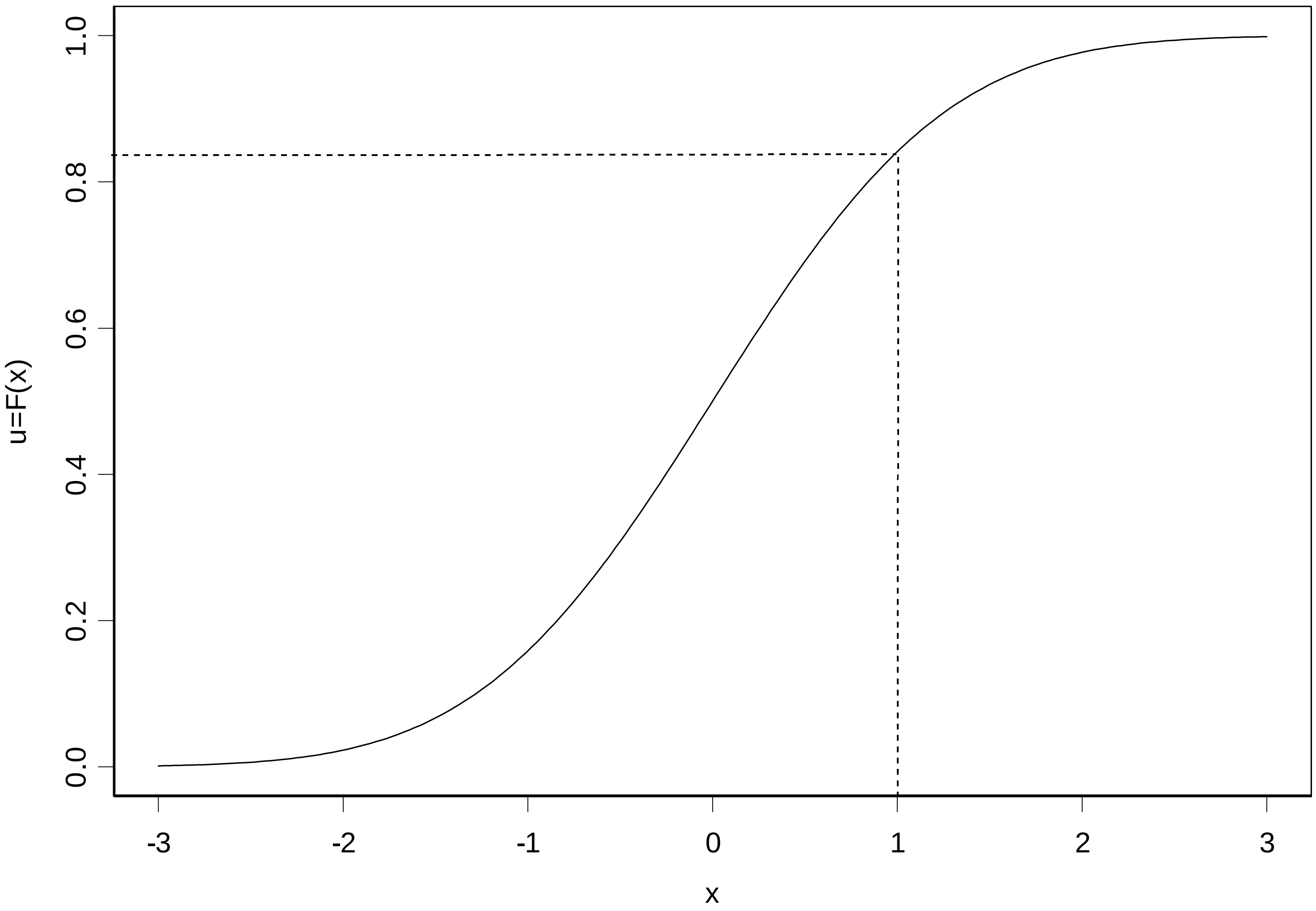
- Suppose we have a portfolio with N assets whose returns follow *arbitrary* univariate marginal distribution functions $F_1(x_1), F_2(x_2), \ldots, F_N(x_N)$. The copula function C combines the marginals to give the joint density such that:

$$C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) = F(x_1, x_2, \ldots, x_N)$$

  $F(x)$ is obviously a uniform random variable

- We define F^{-1} as the *pseudo-inverse* function *i.e.*

$$x = F^{-1}(u) \equiv \sup\{x | F(x) \leq u\}$$



1.0
0.8
0.6
0.4
0.2
0.0
u=F(x)
-3
-2
-1
0
1
2
3
x

- Major use of copula is in the construction of multivariate distributions– *modelling the joint distribution of different risks.*
- Notice that this elementary probability transform is simply the usual approach adopted for simulating data from an arbitrary distribution $F(x)$. In other words generate a random sample from a uniform$[0,1]$ distribution and then apply to each value the inverse function $F^{-1}(u) = x$. The resulting sample will be as if drawn from the distribution $F(x)$.
- Since the multivariate distribution contains *all* the information that exists on *the dependence structure* between the variables the copula contains precisely the same information.
- Moreover since the copula is defined on the transformed *uniform* marginals it contains the information on dependence *irrespective* of the particular marginals of the underlying assets.
- Determine the marginal distribution of each asset- then estimate the copula from the data - contains all the information on both the type and degree of dependence between the assets to determine their *joint distribution* and hence assess *joint risks.*

The Literature

1. **Nelsen R.B.**,(1998) , *An introduction to Copulas*, Lecture Notes in Statistics 139, Springer Verlag.
2. **Joe H.**(1997), *Multivariate Models and Dependence Concepts*, Chapman and Hall, London.
3. **Embrechts, McNeil and Straumann**,*Correlation: Pitfalls and Alternatives*, Imperial College Press
4. **Umberto Cherubini, Elisa Luciano and Walter Vecchiato**,(2004) *Copula Methods in Finance*,Wiley Finance

Many research papers in the last few years

1. **Bouyé E., V.Durrleman, A.Nickeghbali, G.Riboulet and T.Roncalli**, (2000), *Copulas of Finance- A Reading Guide and some Applications*, FERC WP
2. **Embrechts P.,Lindskog F. and A.McNeil (2001) Modelling Dependence with Copula,**

   www.math.ethz.ch/finance
3. **Li D.X**.,(2000), *On Default Correlation: A Copula Function Approach,*Journal of Fixed Income
4. **Cherubini U. and E. Luciano**,(2000) *Value at Risk trade-off and capital allocation with Copulas* , http://web.econ.unito.it/gma/elisa.htm#wps
5. **Cherubini U. and E. Luciano**,(2000) *Multivariate Option Pricing with Copulas*, as above
6. **Frey R. and McNeil A**., (2001)*Modelling Dependent Defaults*, ETH Zurich
7. **Bouyé E., N.Gaussel and M. Salmon,** (2000)*Copulae and Time Dependence*, FERC WP.
8. **Sancetta A. and Satchell S**.,(2001) *Portfolio Construction and Optimisation using Bernstein Approximations to Copulae*, FERC Discussion Paper.
9. **Patton A,** (2000), *Modelling Time Varying Exchange Rate Dependence using Conditional Copula*, UCSD DP

10. **Rockinger M. and Jondeau E.**, (2001), *Conditional Dependency of Financial Series: An Application of Copulas*, Mimeo HEC

11. Credit Lyonnais website http://gro.creditlyonnais.fr/content/rd/rd_math/copulas.htm#debut

...plus Bank of England, UBS Warburg,Credit Lyonnais, HSBC,Deutsche Bank,..........

Examples

Bivariate Gaussian

$$C(u,v;\rho) = \Phi_\rho(\Phi^{-1}(u),\Phi^{-1}(v))$$

with Φ_ρ the bivariate gaussian cdf and Φ^{-1} the inverse of the Gaussian cdf.

$$c(x,y;\rho) = (1-\rho^2)^{-1/2}\exp\{-\frac{1}{2}(1-\rho^2)^{-1}[x^2+y^2-2\rho xy]\}.$$

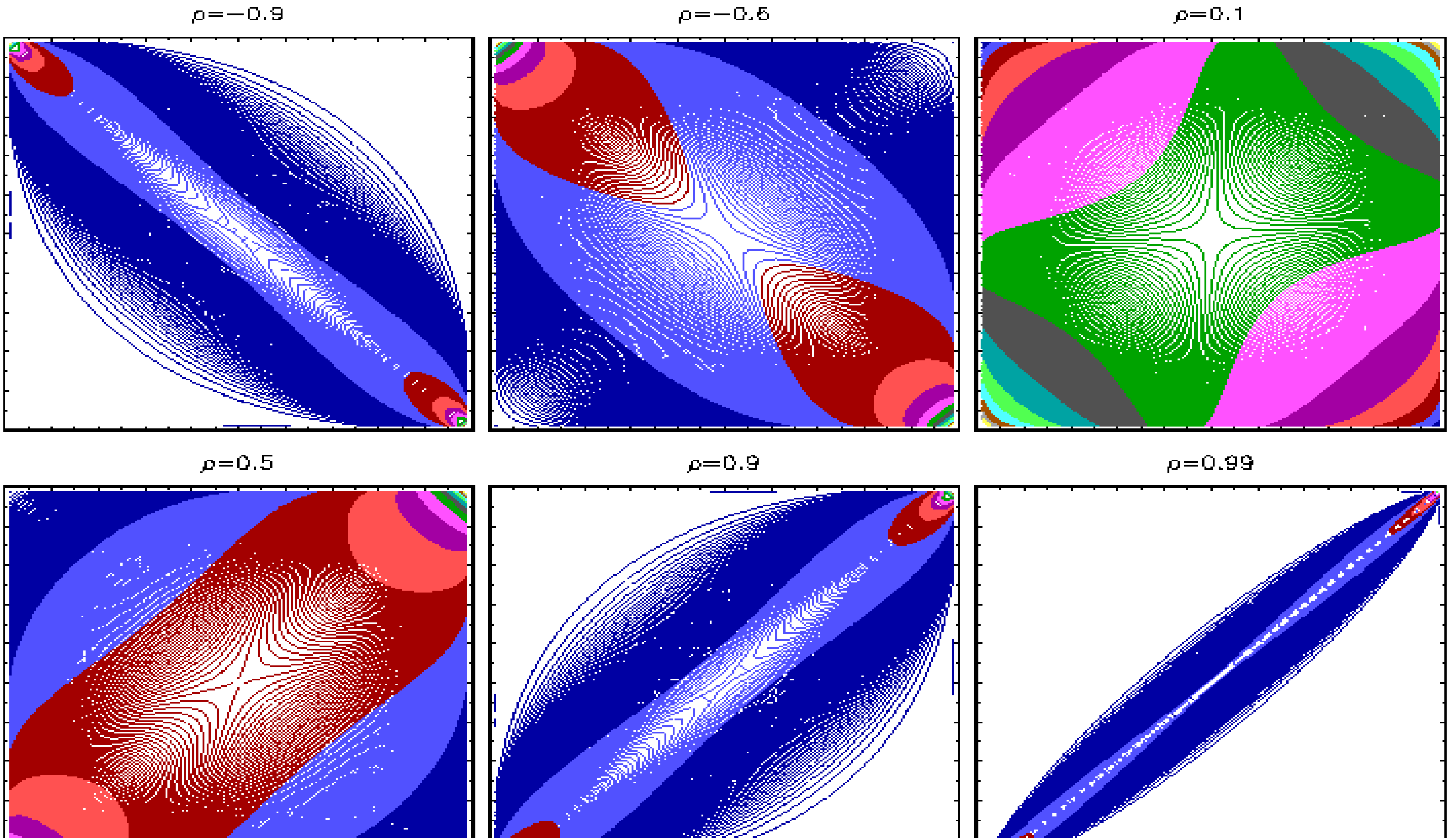
$$\exp\{\frac{1}{2}[x^2+y^2]\}$$

with $x = \Phi^{-1}(u)$ and $y = \Phi^{-1}(v)$

- Dependence measured by the *single* parameter ρ
- Multivariate Gaussian assumption used in mean variance portfolio theory, VaR,... amounts to assuming each asset follows a marginal Gaussian distribution *and* a *Gaussian* Copula.

$C(u,v) = uv \Rightarrow C(.2,.2) = 0.04$

Probability of both FTSE and SP500 being below 20^{th} percentile under independence copula.

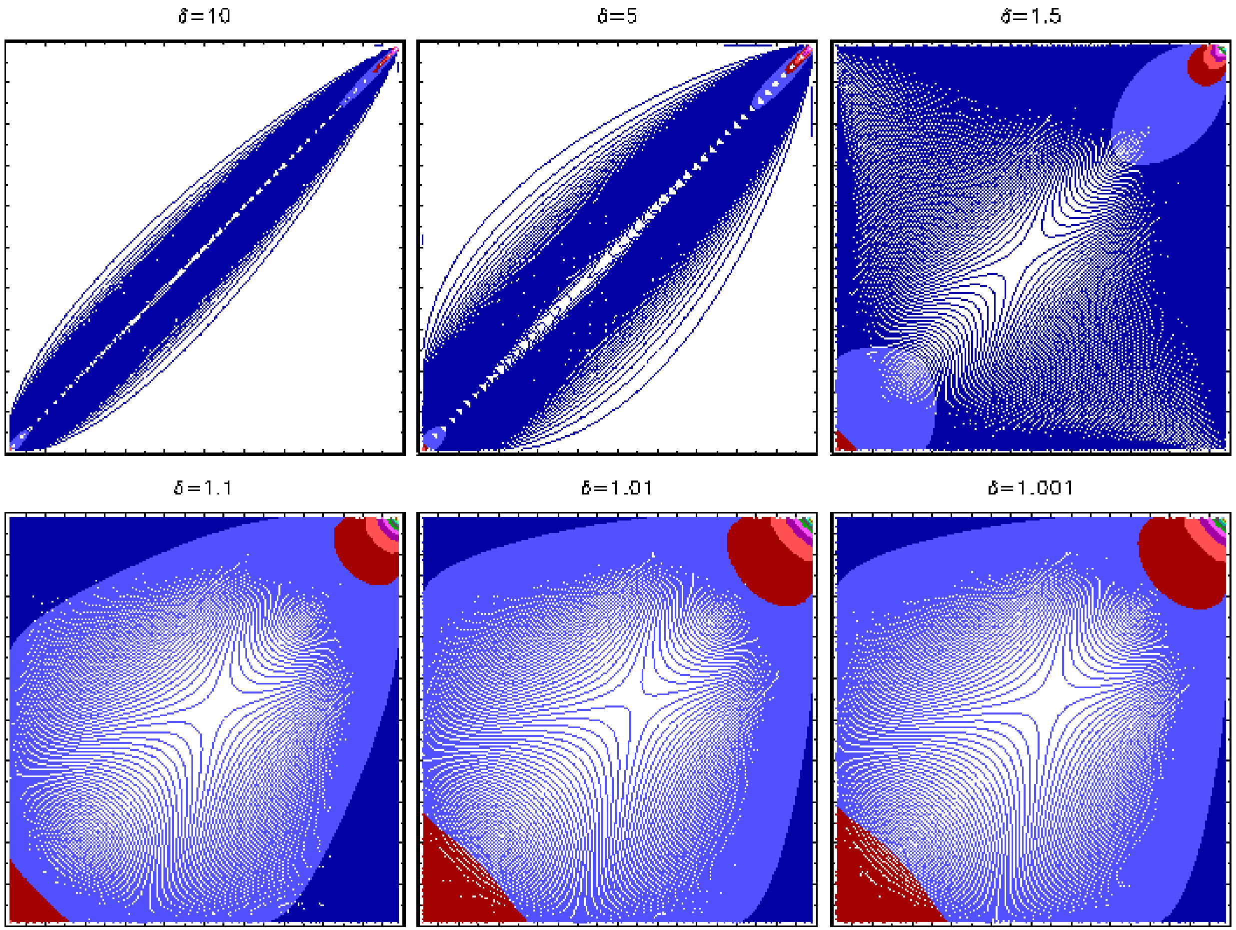

ρ=−0.9
ρ=−0.6
ρ=0.1
ρ=0.5
ρ=0.9
ρ=0.99

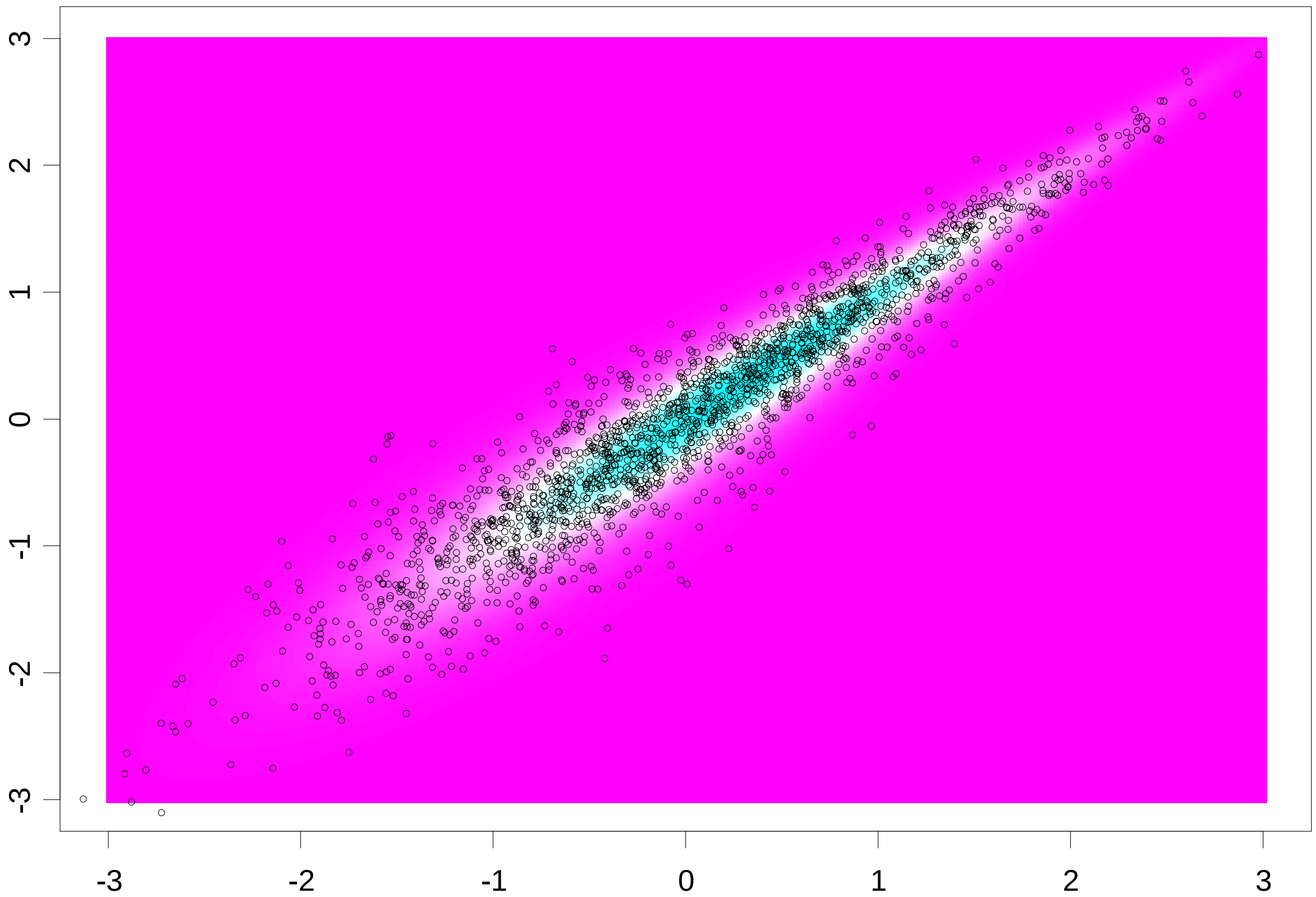
Gumbel(1960)

$$C(u,v;\delta) = \exp\{-((-\log u)^{\delta} + (-\log v)^{\delta})^{1/\delta}\}$$

$\delta = 1$ implies independence and $\delta \to 0$ leads to perfect dependence Increasing dependence at right tails

Gumbel-Hougard Copula $\delta \in (1, \infty)$


δ=10
δ=5
δ=1.5
δ=1.1
δ=1.01
δ=1.001



3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

A two parameter example:

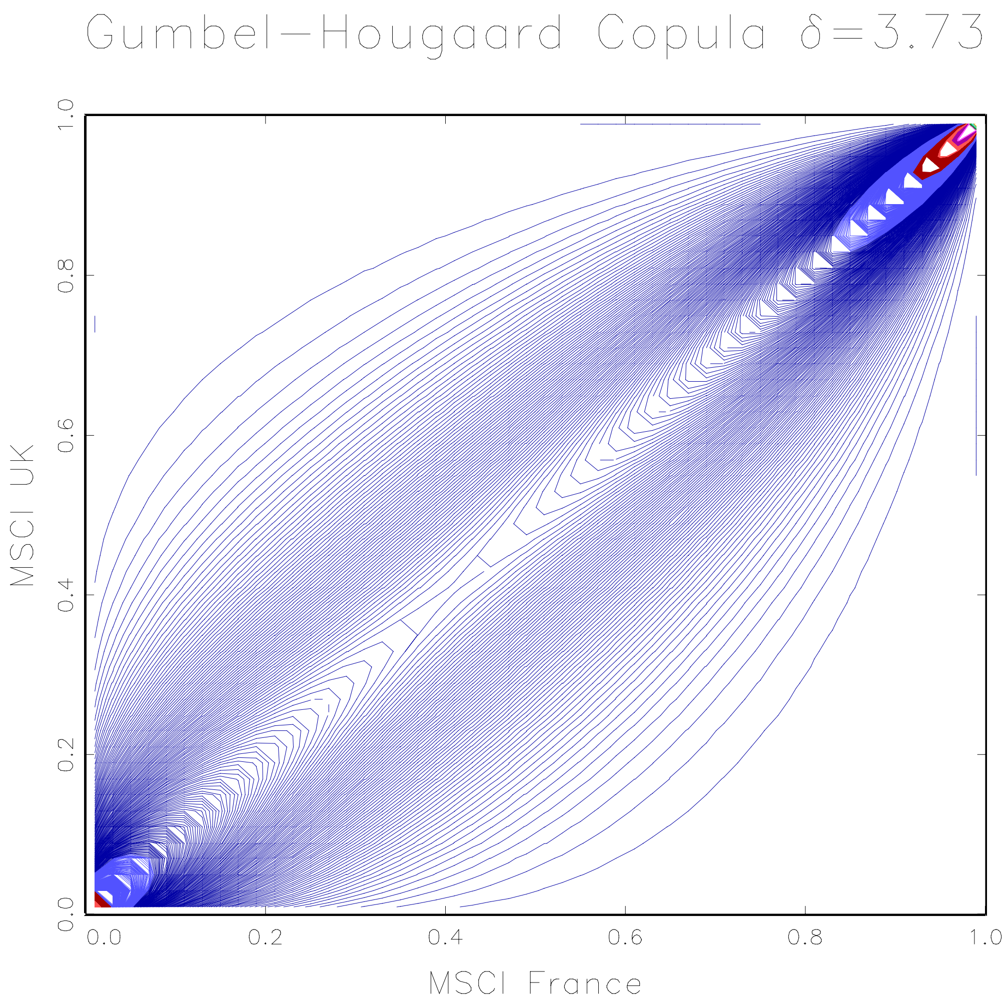
$$C(u,v;\theta,\delta) = \{1 + [(u^{-\theta} - 1)^{\delta} + (v^{-\theta} - 1)^{\delta}]^{\frac{1}{\delta}}\}^{-\frac{1}{\theta}}$$

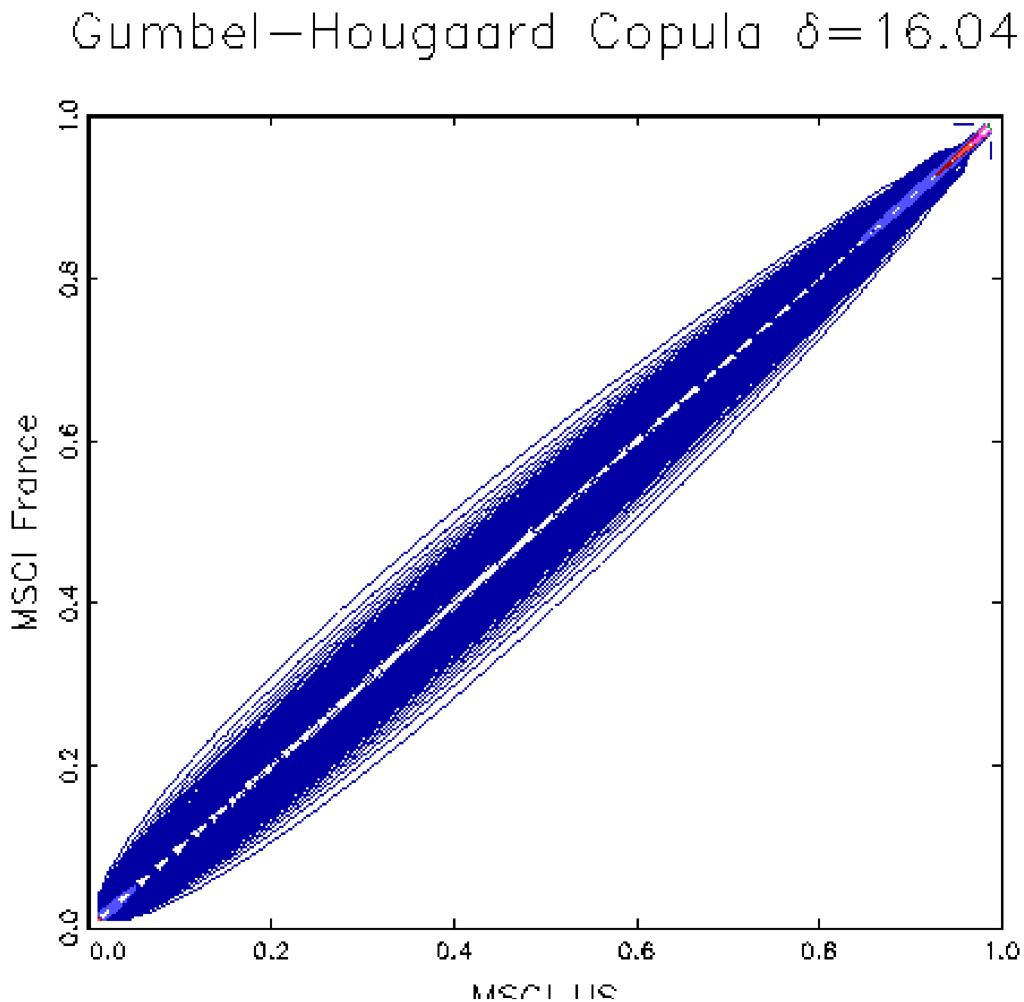
$$= \eta(\eta^{-1}(u) + \eta^{-1}(v))$$

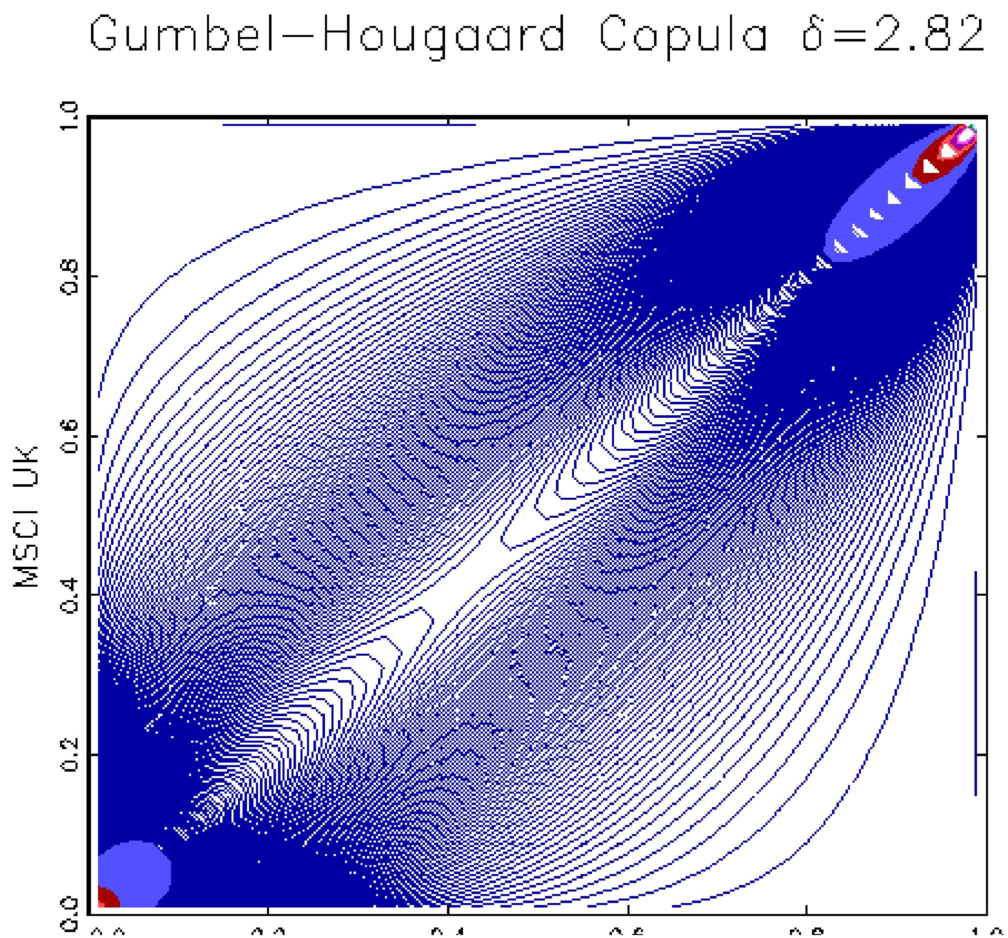
where $\eta(s) = \eta_{\theta,\delta}(s) = (1 + s^{\frac{1}{\delta}})^{-\frac{1}{\theta}}$

- LOWER TAIL DEPENDENCY PARAMETER = $2^{-\frac{1}{\delta\theta}}$
- UPPER TAIL DEPENDENCY PARAMETER = $2\text{-}2^{\frac{1}{\delta}}$ (Independent of θ)
- CONCORDANCE INCREASES as θ INCREASES
- Dependency Measures can usually be expressed in terms of the parameters of the copula.

eg. one parameter for upper tail dependence (δ)and another for concordance(θ)– but the issue is more general– in general dependence measures will be functions of the copula's parameters– although they need not be *the* parameters of the copula.


Gumbel–Hougaard Copula δ=3.73
MSCI UK
MSCI France
0.0
0.2
0.4
0.6
0.8
1.0


Gumbel-Hougaard Copula δ=16.04
MSCI France
MSCI US
0.0
0.2
0.4
0.6
0.8
1.0


Gumbel–Hougaard Copula δ=2.82
MSCI UK
0.0
0.2
0.4
0.6
0.8
1.0

Measuring Dependency

Independence:

If the random variables $X_1, X_2, \ldots\ldots. X_n$ are independent then the copula function that links their marginals is the *product* copula

$$C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) = F(x_1)F(x_2)\ldots, F(x_N) = C^{\perp}$$

- So *tests* for independence can be based on the distance of the empirical copula to the product copula.
- More generally the copula function is defined over the entire range of the random variables (transformed into the uniform$[0,1]$ space) and hence we *can examine the dependence structure through the entire range of the potential variation of the assets behaviour rather than through a single number such as the correlation.*– What are we interested in?

Criteria for Good Dependence Measures

Criteria that any measure of dependence δ between two continuous random variables X_1 and X_2 should satisfy include;

1. δ is defined for every pair (X_1, X_2),
2. $\delta(X_1, X_2) = \delta(X_2, X_1)$, symmetry
3. $-1 \leq \delta(X_1, X_2) \leq 1$,
4. $\delta(X_1, X_2) = 0$ if and only if X_1 and X_2 are independent,
5. $\delta(X_1, X_2) = 1$ if and only if each of X_1 and X_2 is almost surely a strictly monotone function of the other, COMONOTONIC (alternative forms exist ; for instance $\delta(X_1, X_2) = -1$ if and only if each of X_1 and X_2 is almost surely a strictly COUNTER-MONOTONIC function of each other).
6. $\delta(X_1, X_2) = \delta(T_1(X_1), T_2(X_2))$ with T_1 and T_2 almost surely strictly monotone functions,

The Inadequacy of Correlation

Pearson's Correlation Measure

$$\rho = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sigma_x \sigma_y}$$

- Provides a measure of *linear* association
- σ_x^2 and σ_y^2 have to be finite for ρ_{xy} to be defined; eg. extreme value type II (Fréchet) distribution with parameter $\tau = -\alpha^{-1}$ is such that $\int_0^\infty x^r \mathrm{dF}_X(x) = \infty$ for $r > \alpha$. So correlation is not defined in this quite reasonable and important case for financial applications
- Independence *always* implies zero correlation but the converse is only true for a multivariate gaussian (if another joint distribution for gaussian marginals is assumed, the converse does not hold).
- Weak correlations *do not imply* low dependence.
- Correlation is not an *invariant* measure whereas the copula function *is* invariant. $\rho(X,Y) \neq \rho(\log X, \log Y)$

The fundamental reason why correlation fails as a general an invariant measure of dependency is due to the fact that *the Pearson Correlation coefficient depends not only on the copula but also on the marginal distributions*. Thus the measure is affected by changes of scale in the marginal variables.

$$\rho(X,Y) = \frac{1}{\sigma(X)\sigma(Y)} \iint_0^1 [C(u,v) - uv] \underbrace{dF}^{-1}(u) \underbrace{dG}^{-1}(v)$$

Alternative Notions of Dependence

Many different forms of dependence between assets can exist that are simply not captured by correlation; co-skewness, co-kurtosis

for instance

- Patton (2001); Hu (2001) note that stock returns are more dependent during market downturns than during upturns- Sp500, DAX,Nikkei,Hang Seng much higher dependence in Bear Markets than in Bull markets; common sensitivity to bad news stronger than for good news.
- Relative Bear Market not volatility that describes dependence across markets
- asymmetric dependence given positive and negative returns.
- Longin and Solnik(2001); different dependencies for large and small movements in returns.

Measures of Concordance

Functions of the copula will be scale invariant under almost surely strictly increasing transformations$\Rightarrow$ concordance measures are invariant

- Concordance between two random variables arises if large values of one variable arise with large values of the other and small values occur with small values of the other

Kendall's Tau

If (X_1, Y_1) and (X_2, Y_2) are independent and identically distributed random vectors with possibly different *joint* distribution functions H_1, and H_2 with copulae C_1 and C_2 respectively but with common margins. The population version of Kendall's tau is defined as the probability of concordance minus the probability of discordance, Q

$$\tau = \tau_{X,Y} = [P(X_1 - X_2)(Y_1 - Y_2) > 0] - [P(X_1 - X_2)(Y_1 - Y_2) < 0]$$

Nelsen shows that this may be re-expressed in terms of the copulae as

$$Q = Q(C_1, C_2) = 4 \iint_{I^2} C_2(u,v)\,dC_1(u,v) - 1$$

Spearman's Rho

Let $\mathbf{R_i}$ be the rank of $\mathbf{x_i}$ among the **x's** and $\mathbf{S_i}$ be the rank of $\mathbf{y_i}$ among the **y's.** The Spearman rank order correlation coefficient is:

$$\rho_S = \frac{\sum_{i=1}^{n}(R_i - \bar{R})(S_i - \bar{S})}{(\sum_{i=1}^{n}(R_i - \bar{R})^2)^{½}(\sum_{i=1}^{n}(S_i - \bar{S})^2)}$$

which again my be expressed in terms of copulae as

$$\rho_C = 12\int\int_{[0,1]^2}(C(u,v) - uv)dudv$$

Spearman's rank correlation coefficient is essentially the ordinary correlation of $\rho(F_1(X_1), F_2(X_2))$ for two random variables $X_1 \sim F_1(.)$ and $X_2 \sim F_2(.)$.

- Essentially these two measures of concordance measure the degree of *monotonic* dependence as opposed to the Pearson Correlation which simply measures the degree of *linear* dependencei
- They both achieve a value of unity for the bivariate Fréchet upper bound (one variable is an increasing transformation of the other) and minus one for the Fréchet lower bound (one variable is strictly decreasing transform of the other). Functional dependence as opposed to linear dependence.

Daily log-returns for MSCI US, MSCI France and MSCI UK from 12/1987 to 12/1999.

Dependence Measures	Kendall	Spearman	Gini	Correlation
MSCI US - MSCI France	0.156	0.228	0.564	0.168
MSCI US - MSCI UK	0.180	0.264	0.604*	0.358
MSCI UK - MSCI France	0.397*	0.557*	0.470	0.527*

NB. Concordance may also be zero even if variables are dependent but bounded between 0 and 1 regardless of marginal distributions.

Gumbel Copula indicated MSCI France and US were most highly associated in terms of δ for which high values suggests *tail* dependence important.

Positive Quadrant Dependent (PQD)

$$\Pr\{X > x, Y > y\} \geq \Pr\{X > x\} \Pr\{Y > y\}$$

So probability that two assets make large gains is greater than if they were independent or in terms of copula

$$C > C^{\perp}$$

Survival Copulae

- Key role in credit risk management is the class of *survival copulae*. Assume two risks A and B with their respective survival times represented by two random variables T_A and T_B.
- Their survival functions are given by $S_A(t_A) = 1 - F_A(t_A) = \Pr\{T_A > t_A\}$ and $S_B(t_B) = 1 - F_B(t_B) = \Pr\{T_B > t_B\}$.
- Let C be the copula that links T_A and T_B, the joint density of the times to default of two risks. Then the joint survival function:

$$\begin{aligned} S(t_A, t_B) &= \Pr\{T_B > t_B, T_B > t_B\} \\ &= S_A(t_A) - S_B(t_B) - 1 + C(1 - S_A(t_A), 1 - S_B(t_B)) \end{aligned}$$

Defining $\tilde{C}(u,v) = u + v - 1 + C(1-u, 1-v)$ as the *Survival Copula* of T_B and T_A and we have

$$S(t_A, t_B) = \tilde{C}(S_A(t_A), S_B(t_B))$$

$\tilde{C}$ "couples" the joint survival function to its univariate marginals and provides the means of addressing *the joint default risk.*

Tail Area Dependence and Extremes

Environmental Science has developed an empirical dependence measure for extremes, λ - so called *tail dependence* where asymptotic independence is given by $\lambda = 0$ and $\lambda \in (0,1]$ for upper tail dependence.

λ_u is linked to the asymptotic behaviour of the copula:

$$\lambda_u = \lim_{\alpha \to 1^-} \Pr\{X_2 > VaR_\alpha(X_2) \mid X_1 > VaR_\alpha(X_1)\}$$

$$= \lim_{\alpha \to 1^-} \frac{1 - 2\alpha + \bar{C}(\alpha, \alpha)}{1 - \alpha}$$

NB.the tail area dependency measure λ_u depends on the copula and not on the marginal distributions.

- Quantile based measures of extreme dependence look highly promising tools for risk management.

Choice between Copula:

- Different copula exhibit different dependency structures in different parts of the potential range of their margins, for instance
  - the *gaussian copula* implies when $\rho \neq 1$that the variables are asymptotically independent, ie. $\lambda_u = 0$ for $\rho < 1$
  - whereas the *t copula* implies extremes are asymptotically dependent for $\rho \neq -1$.
  - The *Frank copula* implies a *symmetric* dependence pattern, ie. the dependence is the same between positive returns as between negative returns.
  - *Clayton copula* implies higher dependence in bear markets.
  - *Gumbel copula* implies higher dependence in bull markets, increasing dependence in right tails so used to model extremes.
  - Asymmetric Dependence– becoming another clear stylised fact
- need careful empirical selection,
  - Standard approach: Goodness of Fit, AIC often used to select between parametric copula or non-parametric copula used.
  - Multivariate Encompassing Tests; Salmon (2002) Simulation based non-nested tests

Financial Applications

Essentially 4 main areas:

1. Option Pricing
2. Credit Risk Modelling
3. Risk management
4. Measuring different forms of dependence and using them to construct investment strageies

- *Li* at Riskmetrics is using Survival Copulae to measure *default dependency*; standard Riskmetrics approach implied the use of Gaussian Copula. Recently extended by *Frey and McNeil* (2001)
- *Hwang and Salmon* (2000) use copulae to capture the relationship between different *performance measures*, VaR and Tracking Error.
- *Cherubini and Luciano* (2001a,2002b) consider *option pricing* and *VaR*. Options based on multidimensional underlying. *Rosenberg* (1999)(2000) pricing multivariate contingent claims.
- *Bouyé and Salmon* (2000) are developing dynamic, nonlinear *quantile based risk models* using Copulae, cf. CAViaR (Engle and Manganelli, UCSD WP)

- Portfolio design, *Patton* (2001), *Sancetta and Satchell*(2001)
- *Kat and Salmon* (2002) looking at forms of dependency between hedge funds and market indices.
- *Aris Bikos* (2001),(Bank of England) uses copulae to construct *multivariate implied pdfs*-drawn from the option markets

  risk neutral copula.
- *Credit-Lyonnais*(1999-..) doing everything- pricing credit derivatives,VaR bounds...

Copula Quantile Regression

- Extend regression dependence to *quantile* regression with structural form determined by

  the copula

Why?

- examine dependence at moderate and extreme levels
- Nonlinear Pairs trading (loss aversion)

Why?

- Many equilibrium relationships between assets will not necessarily be through the conditional expectation

cf. **Salmon and Sarno**,**(2004), Nonlinear Pairs Trading; The Forward Premium and Loss Aversion , Warwick FERC DP**

Dynamic Copula Quantile Regressions

cf. **Bouye and Salmon (2003), Dynamic Copula Quantile Regression and Tail area Dependence, Warwick FERC DP**

- Define the probability distribution of *y conditional* on *x* by $p(x, y; \delta)$:

$$
\begin{aligned}
p(y|x;\delta) &= \Pr\{Y \le y \mid X = x\} \\
&= \mathbf{E}(\mathrm{I}_{\{Y\le y\}} \mid X = x) \\
&= \lim_{\varepsilon\to 0} \Pr\{Y \le y \mid x \le X \le x+\varepsilon\} \\
&= \lim_{\varepsilon\to 0} \frac{\mathbf{F}(x+\varepsilon,y;\delta) - \mathbf{F}(x,y;\delta)}{F_X(x+\varepsilon) - F_X(x)} \\
&= \lim_{\varepsilon\to 0} \frac{\mathbf{C}[F_X(x+\varepsilon),F_Y(y);\delta] - \mathbf{C}[F_X(x),F_Y(y);\delta]}{F_X(x+\varepsilon) - F_X(x)}
\end{aligned}
$$

$$p(y|x;\delta) = C_1[F_X(x),F_Y(y);\delta]$$

with $C_1(u,v;\delta) = \frac{\partial}{\partial u}C(u,v;\delta)$. Since the distribution functions F_X and F_Y are nondecreasing, $p(y|x;\delta)$ is nondecreasing in y. Using the same argument, $p(y|x;\delta)$ is nondecreasing in x if $C_{11}(u,v;\delta) \le 0$ and nonincreasing in x if $C_{11}(u,v;\delta) \ge 0$ where $C_{11}(u,v;\delta) = \frac{\partial^2 \mathbf{C}(u,v;\delta)}{\partial u^2}$.

Definition *For a parametric copula $\mathbf{C}(.,.;\delta)$, the **p-th copula quantile curve** of y conditional on x is defined by the following implicit equation*

$$p = C_1[F_X(x),F_Y(y);\delta]$$

where $\delta \in \Delta$ the set of parameters.

- Solving thequantile relationship between X and Y:

$$y = \mathbf{q}(x, p; \delta)$$

where $\mathbf{q}(x, p; \delta) = F_Y^{[-1]}(D(F_X(x), p; \delta))$ with D the partial inverse in the second argument of C_1 and $F_Y^{[-1]}$ the pseudo-inverse of F_Y.

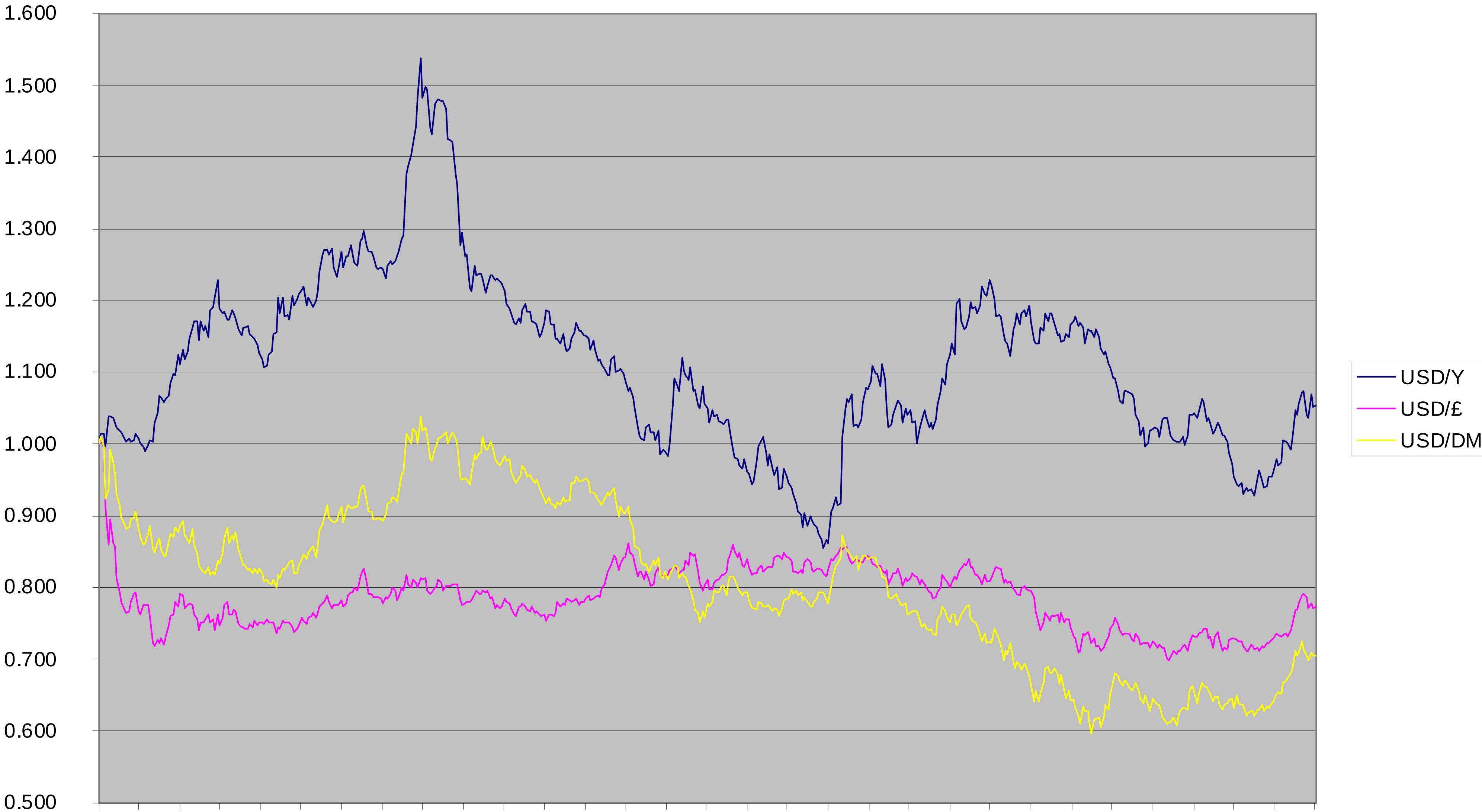
Note that the relationship can alternatively be expressed using uniform margins as:

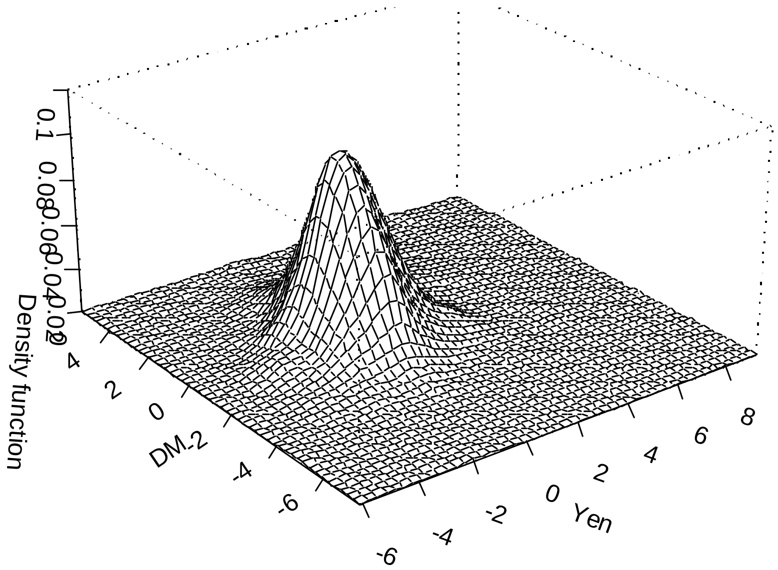
$$v = \mathbf{r}(u, p; \delta).$$

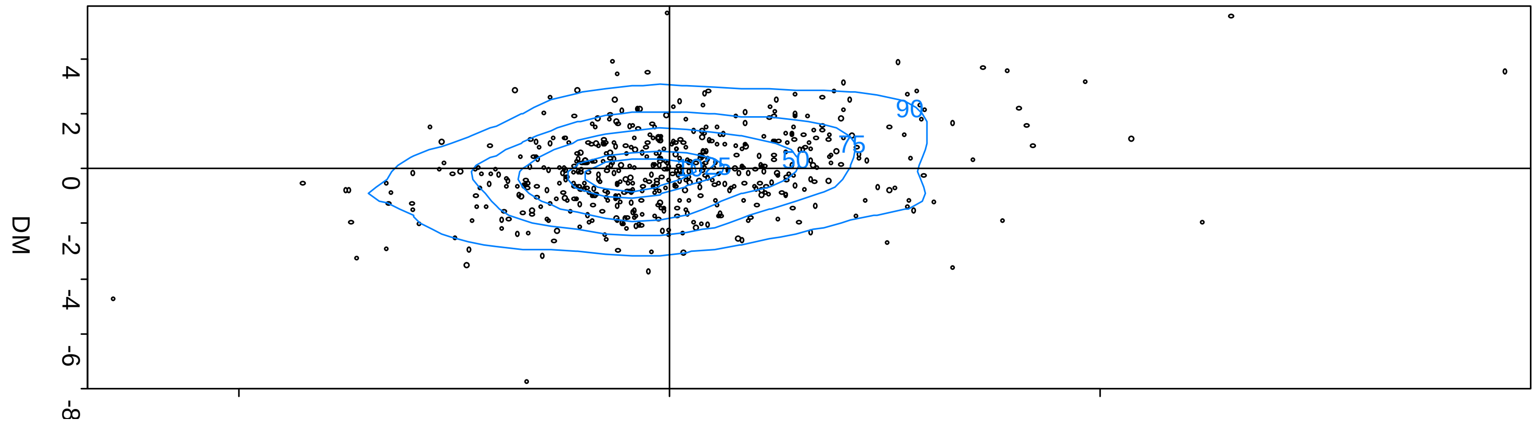
with $u = F_X(x)$ and $v = F_Y(y)$.

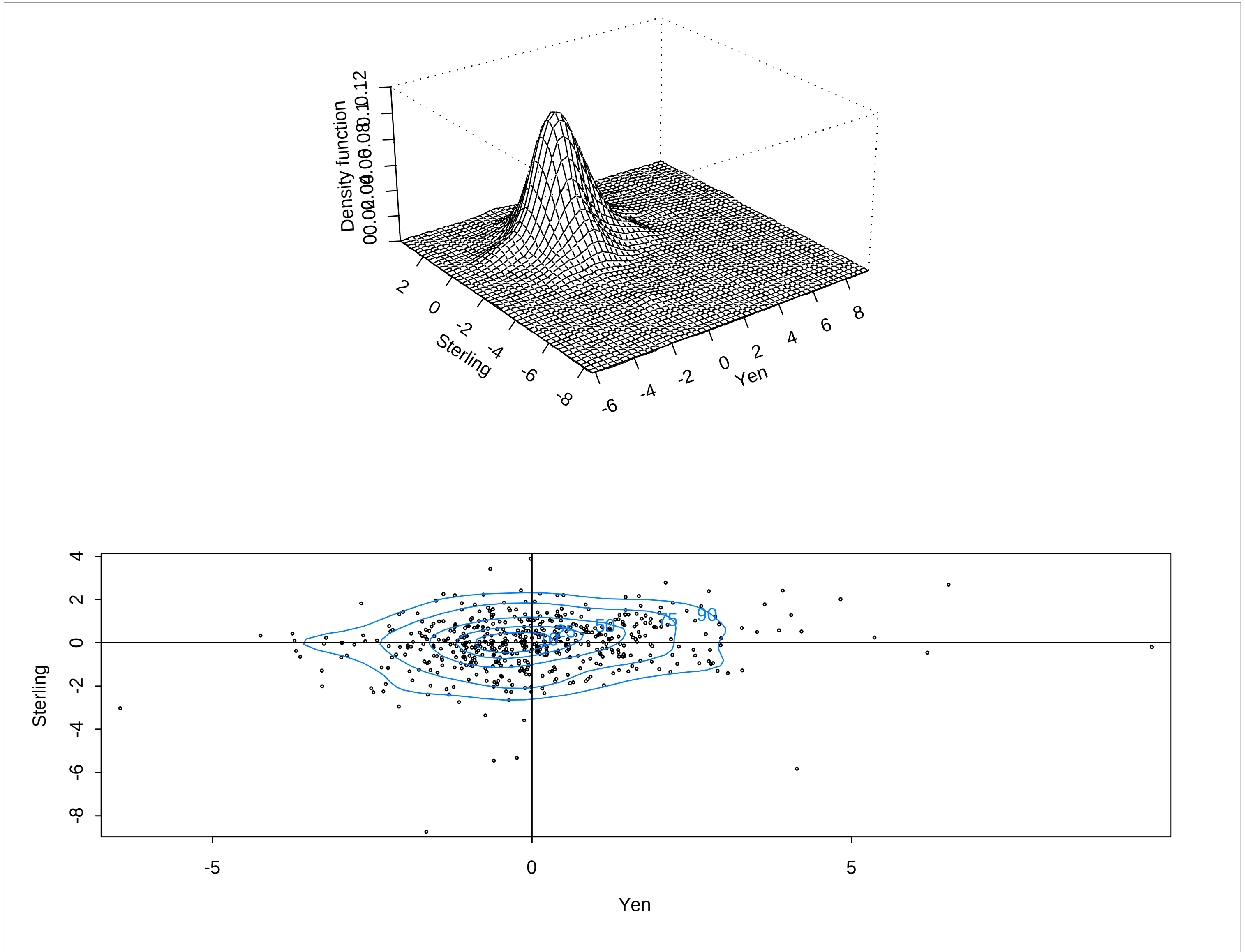
Application to FX markets

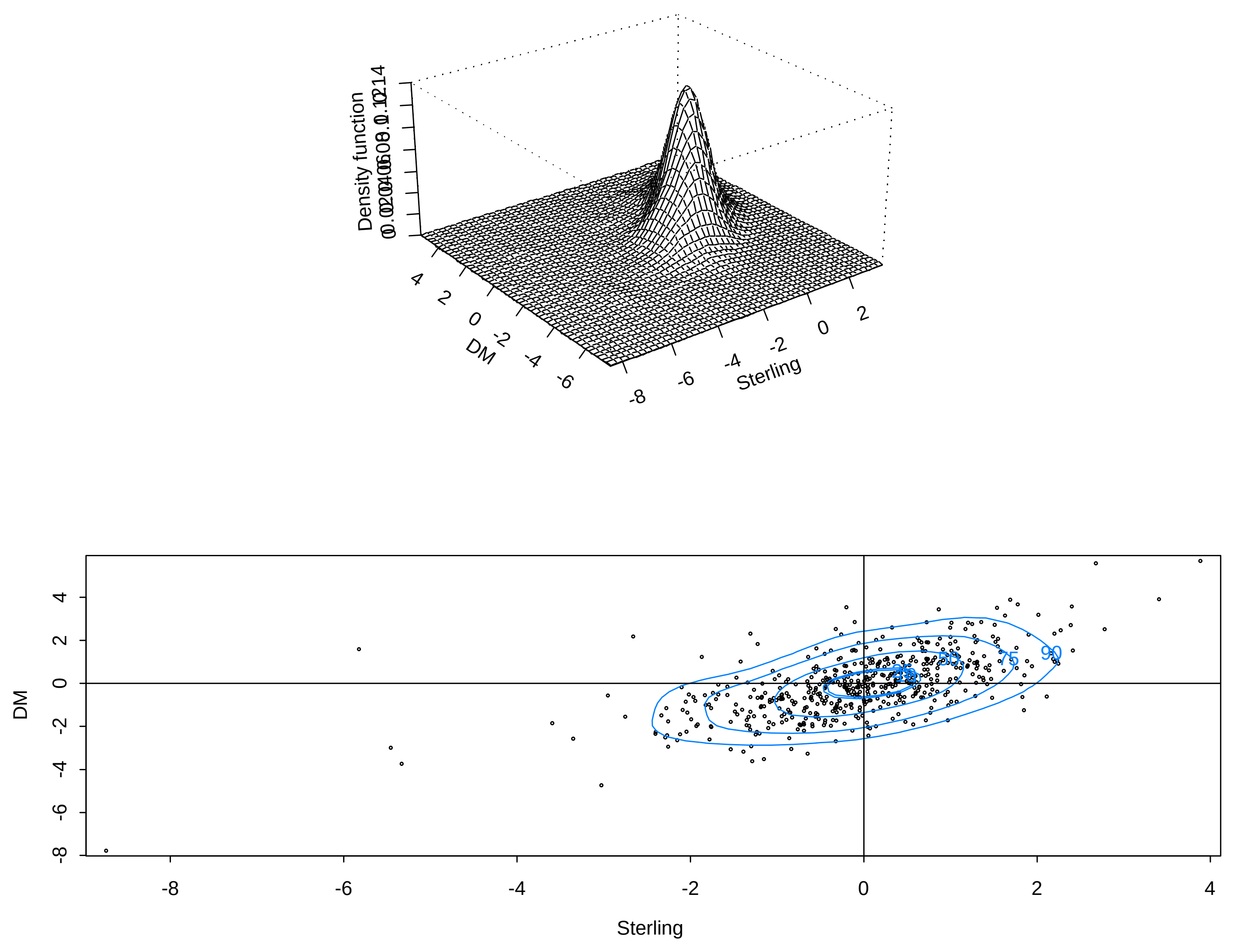
- Dollar -Yen, Dollar-Sterling and Dollar- DM rates using 522 *weekly* returns from August 1992 to August 2002;


1.600
1.500
1.400
1.300
1.200
1.100
1.000
0.900
0.800
0.700
0.600
0.500
USD/Y
USD/£
USD/DM


Density function
0.1
DM.2
Yen
4
2
0
-2
-4
-6
8
6


DM
4
2
0
-2
-4
-6
-8
90
75
50
25


Density function
0.12
Sterling
Yen
50
75
90


Density function
0.02
0.04
0.06
0.08
0.10
0.12
0.14
DM
Sterling
Sterling
DM
25
50
75
90

We compute the nonlinear quantile regression estimates of $\hat{\rho}(p)$ such that:

$$\hat{\rho}(p) = \arg\min\left(\sum_{t=1}^{T}\left(p - \mathrm{I}_{\{S_{1t}\leq \mathbf{q}(S_{2t},p;\rho,\hat{\theta}_1,\hat{\theta}_2)\}}\right)\left(S_{1t} - \mathbf{q}\left(S_{2t},p;\rho,\hat{\theta}_1,\hat{\theta}_2\right)\right)\right)$$

Assuming a Gaussian Copula the relationship between any two exchange rates S_1 and S_2 at the p'th-quantile is:

$$S_1 = \hat{F}_1^{[-1]}\left[\Phi\left(\hat{\rho}(p)\Phi^{[-1]}\left(\hat{F}_2(S_2)\right) + \sqrt{1-\hat{\rho}^2(p)}\,\Phi^{[-1]}(p)\right)\right], \qquad \#$$

with $\hat{F}_1$ and $\hat{F}_2$ the *empirical* marginal distribution functions for the two exchange rates. The estimates of the copula parameter (which in this case is just the correlation coefficient) at each quantile level $\hat{\rho}(p)$, expressed in percentage terms, are reported in Tables[1] and [2] below together with their estimated standard deviations. The mean regression results are also reported for information. The lower p the higher the quantile regression curve.

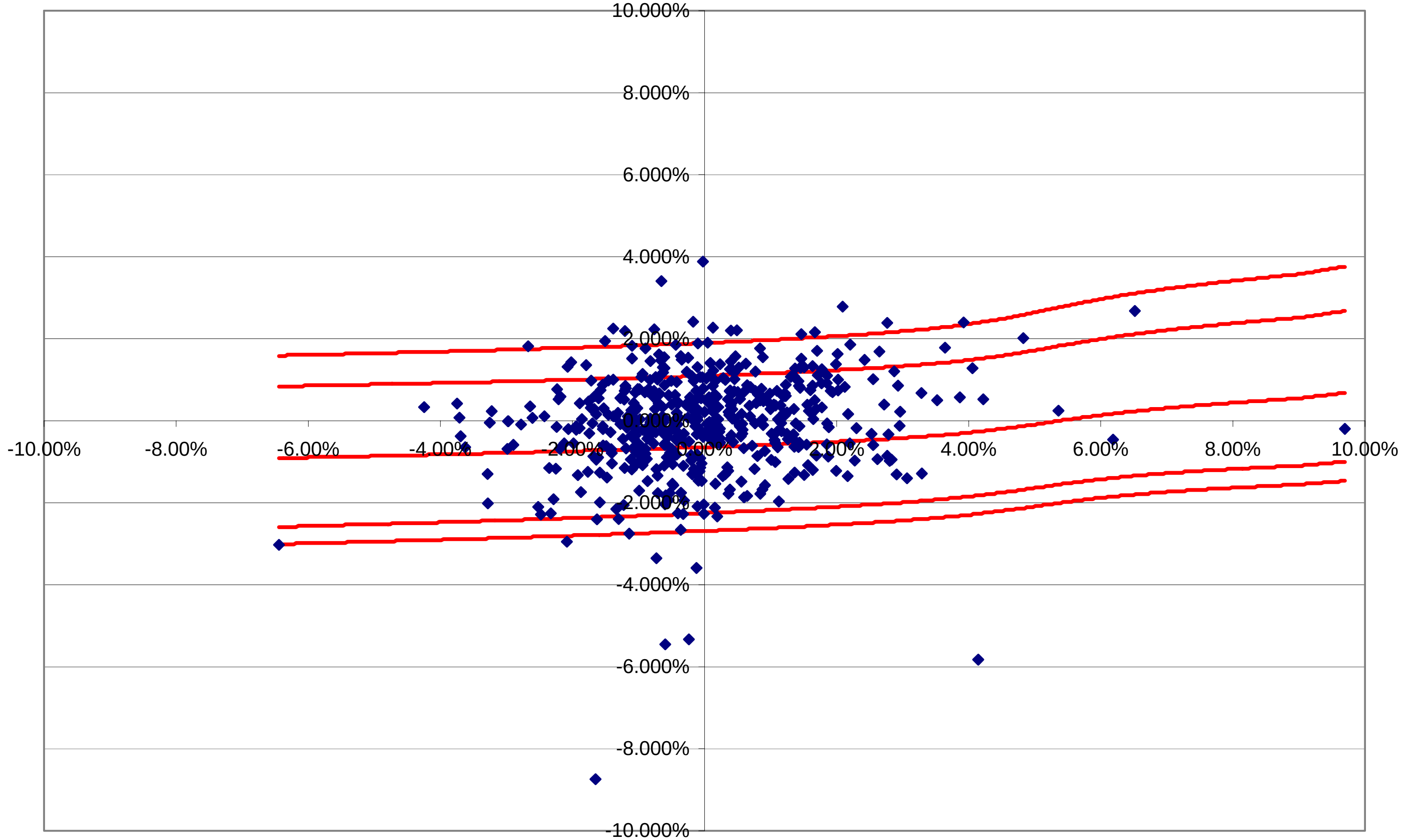
S_1	**USD/Y**	**USD/Y**	**USD/£**
S_2	**USD/£**	**USD/DM**	**USD/DM**
5%	14.2	37.7	49.1
	(5.4)	*(3.5)*	*(4.6)*
10%	16.5	31.9	57.2
	(4.7)	*(4.2)*	*(4.0)*
50-%	20.2	32.9	72.0
	(3.8)	*(4.0)*	*(3.1)*
90%	14.1	28.5	63.2
	(5.5)	*(4.7)*	*(3.6)*
95%	13.2	23.3	55.8
	(5.9)	*(5.7)*	*(4.1)*
mean regression	18.3	32.0	65.2
	(4.2)	*(4.2)*	*(3.5)*

S_1	**USD/£**	**USD/DM**	**USD/DM**
S_2	**USD/Y**	**USD/Y**	**USD/£**
5%	14.4	21.4	51.2
	(5.9)	*(6.2)*	*(4.0)*
10%	17.5	20.1	57.7
	(4.9)	*(6.6)*	*(3.6)*
50%	20.5	33.4	64.3
	(4.1)	*(4.0)*	*(3.2)*
90%	22.8	37.1	66.1
	(3.7)	*(3.6)*	*(3.1)*
95%	16.9	34.3	51.2
	(5.0)	*(3.9)*	*(4.0)*
mean regression	19.2	32.0	62.1
	(4.4)	*(4.2)*	*(3.3)*

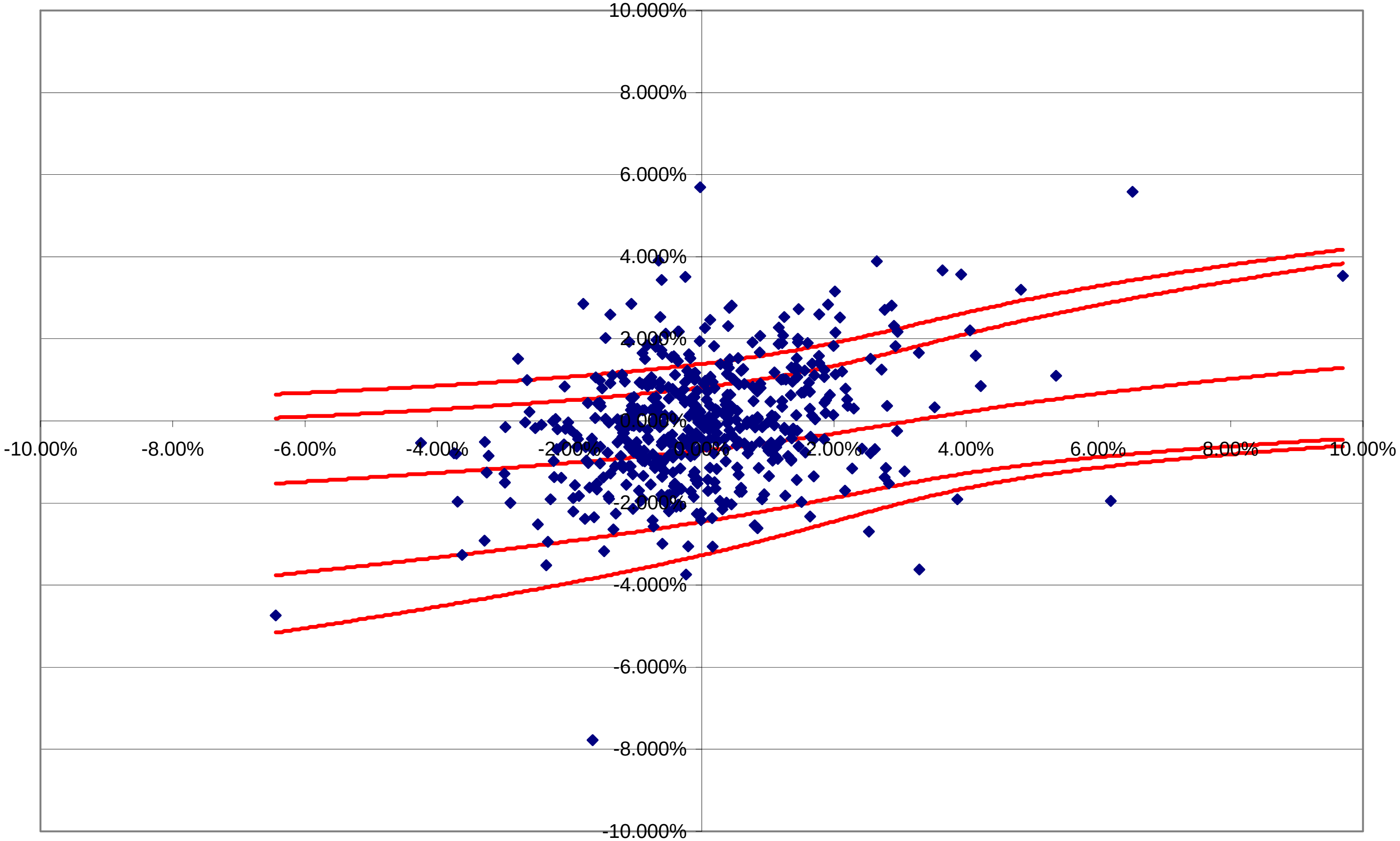
r_t	**USD/Y**		**USD/Y**		**USD/£**	
r_t	**USD/£**		**USD/DM**		**USD/DM**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.07	0.00	1.17*	0.00	1.42*	0.19*
10%	1.07	0.00	1.17*	0.00	1.39*	0.21*
50%	1.06	0.03	1.13	0.09	1.21*	0.37*
90%	1.05	0.08	1.10	0.21	1.07	0.53*
95%	1.04	0.09	1.09	0.23	1.06	0.55*

- We briefly compare these Gaussian Copula results with those from using the Joe-Clayton Copula where the stars indicate significance at the 95% level from the value of one for θ (upper tail dependency) and zero for δ. (lower tail dependency).
- We can see the same indication of upper tail dependence in the Yen:DM dollar rates in levels and Sterling:DM dollar rates in the upper tail in returns.
- Some lower tail dependence is found for the Yen:Sterling Dollar rates and Sterling DM Dollar rates in levels and more strongly in the Sterling: DM in returns. Otherwise we find little or no dependence at all with $\hat{\theta}(p)$ being approxiamtely 1 and $\hat{\delta}(p)$ not significantly different from 0 for most quantile levels.

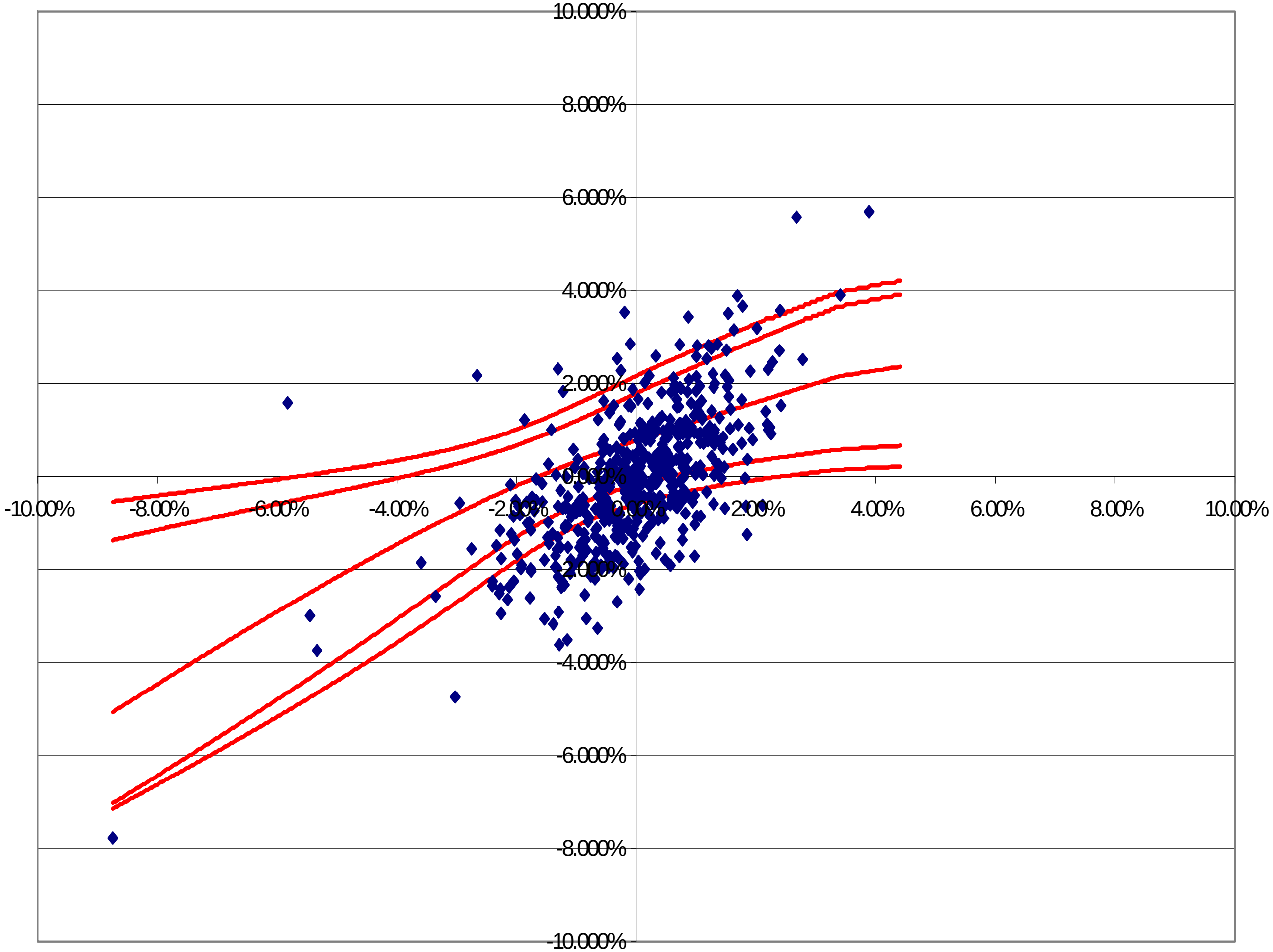
- The obvious advantage from using the Joe Clayton copula is that we can separate the dependence parameters θ and δ with their distinct interpretations from the correlation which describes the entire dependence structure with the Gaussian Copula .



Nonlinear quantile regression of *USD/Y* on *USD/£* for 5%, 10%, 50%, 90%,



Nonlinear quantile regression of *USD/Y* on *USD/DM* for 5%, 10%, 50%, 90



Nonlinear quantile regression of *USD*/£ on *USD*/*DM* for 5%, 10%, 50%, 90%

Dynamic c-quantiles

We next compute the nonlinear *dynamic* quantile regression estimates $\left(\hat{\delta}(p),\hat{\theta}(p)\right)$ using the Clayton Joe Copula concentrating now only on the weekly return data so that:

$$\left(\hat{\delta}(p),\hat{\theta}(p)\right) = \arg\min\left(\sum_{t=1}^{T}(p - \mathrm{I}_{\{r_t \le \mathbf{q}(r_{t-1},p;\delta,\theta)\}})(r_t - \mathbf{q}(r_{t-1},p;\delta,\theta))\right)$$

with

$$\mathbf{q}(r_{t-1},p;\delta,\theta) = \hat{F}^{[-1]}\left[\phi_{\delta,\theta}^{-1}\left[\phi_{\delta,\theta}\left(\phi_{\delta,\theta}{}'^{-1}\left(\frac{1}{p}\phi_{\delta,\theta}{}'\left(\hat{F}(r_{t-1})\right)\right)\right) - \phi_{\delta,\theta}\left(\hat{F}(r_{t-1})\right)\right]\right]$$

with $\phi_{\delta,\theta}$ the generator of the copula defined above and $\hat{F}$ the empirical distribution function of the exchange rate return r_t. The estimates are given in Tables below:

r_{t-1}	**USD/Y**		**USD/£**		**USD/DM**	
r_t	**USD/Y**		**USD/Y**		**USD/Y**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.00	0.00	1.00	0.00	1.00	0.00
10%	1.01	0.00	1.00	0.00	1.00	0.00
50%	1.03*	0.00	1.00	0.00	1.00	0.00
90%	1.05*	0.00	1.01	0.00	1.01	0.00
95%	1.05*	0.00	1.01	0.00	1.02	0.00

r_{t-1}	**USD/Y**		**USD/£**		**USD/DM**	
r_t	**USD/£**		**USD/£**		**USD/£**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.02	0.00	1.00	0.08	1.00	0.06
10%	1.02	0.00	1.00	0.07	1.00	0.06
50%	1.02	0.00	1.00	0.01	1.00	0.04
90%	1.02	0.00	1.00	0.00	1.02	0.00
95%	1.02	0.00	1.00	0.00	1.03*	0.00

r_{t-1}	**USD/Y**		**USD/£**		**USD/DM**	
r_t	**USD/DM**		**USD/DM**		**USD/DM**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.00	0.02	1.00	0.08	1.00	0.06
10%	1.00	0.02	1.00	0.07	1.00	0.06
50%	1.00	0.02	1.00	0.04	1.00	0.04
90%	1.00	0.01	1.00	0.00	1.02	0.00
95%	1.00	0.01	1.00	0.00	1.02	0.00

- These results show that there is no significant dynamic dependence, either cross rates or within rates, at *any* quantile level between the returns of the exchange rates in this weekly data.
- The Clayton Joe parameter estimates indicate independence even in the relative extremes of the joint distribution. This result appears to suggest that forex markets retain efficiency, in a very standard sense, even when the markets are in crisis and in either the upper or lower tail.

Conclusions

- In many ways our life has just become much more *difficult*– We need to be absolutely clear about the form of dependency we are interested in measuring– around the mean, in the tails, PQD etc and a range of new dependency measures will be developed.
- Correlation can often tell very little about the relevant dependency pattern
- A huge range of potential applications are now possible, removing the multivariate Gaussian assumption– pricing etc approximations much better reflection of true underlying distributions and dependencies.
- Role of fat tailed *elliptic* distributions as opposed to Gaussian needs to be re-evaluated– retain mean variance analysis- implies *regime detection* and analysis of time varying skewness models of *Autoregressive Conditional Skewness* and *CaVaR.*
- A number of statistical issues need to resolved, in particular estimation of multivariate copula and the methods to discriminate formally between competing copulae.
- Critical importance of the general-specific approach to model building provided by copulae. Volatlity functions determined by Copulae.
- Copulae provide a practical approach to solving a number of important issues in finance